OBSERVATIONS

ON

REVERSIONARY PAYMENTS:

ON

SCHEMES FOR PROVIDING ANNUITIES
FOR WIDOWS, AND FOR PERSONS IN OLD AGE:

ON

THE METHOD OF CALCULATING THE VALUES OF
ASSURANCES ON LIVES,

AND ON

THE NATIONAL DEBT.

ALSO,

ESSAYS on different Subjects in the Doctrine of Life-Annuities and Political Arithmetic;

A Collection of New Tables, and a Postscript on the Population of the Kingdom.

BY RICHARD PRICE, D.D. F.R.S.

IN TWO VOLUMES.

THE WHOLE NEW ARRANGED, AND ENLARGED BY THE
ADDITION OF ALGEBRAICAL AND OTHER NOTES,
AND THE SOLUTIONS OF SEVERAL NEW PROBLEMS IN THE
DOCTRINE OF ANNUITIES,

BY WILLIAM MORGAN, F.R.S.

SEVENTH EDITION.

VOL. I.

LONDON
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INTRODUCTION

TO THE

SEVENTH EDITION.

IN the present edition I have neither altered the arrangement, nor added to the matter of this work; having done all that I thought necessary in those respects in the former edition.—The different Societies which were formed in London about forty years ago, and whose defective plans are so justly exposed and condemned in the first Volume, have, with very few exceptions, sunk into insignificance or ruin; and the account of them is no further interesting than as it constitutes an original part of this work, and may perhaps serve to prevent the formation of similar Societies in future.—The great success and flourishing state of the Equitable Society have led to the establishment of numerous other offices for making assurances on lives and survivorships;—in all of which the tables of that Society have been adopted with little or no variation:—But in most of those offices the profits, instead of being equally divided, as in the Equitable Society, among the *assured*, are to be distributed exclusively among the *assurers*, in proportion to the sums which they have subscribed towards forming the capital of the company. I am sorry, however, to observe, that I know of no society or company of this kind whose plan is altogether unexceptionable.—The original constitution of the Equitable Society was formed on principles which must long ago have produced its ruin, had it not been timely checked and improved by the prudent forbearance and discretion of succeeding members. By the measures taken to guard against the improvident distribution of its stock, and, as far as circumstances will admit, to make every person to participate in proportion to his interest in such part of the surplus as may be safely alienated from its capital, this Society cannot well be made more secure, or rendered a greater benefit to the public.—But it should be remembered (and it behoves the new Assurance Companies to bear it particularly in mind) that the premiums of this Society were originally twice as high as they are at this time—that they had existed 20 years before those premiums were in the least reduced, and above 30 years before they were reduced to their present amount—that the greater part of their assurances at first were for short terms (which for many reasons are by far the most profitable)—that their expenditure was very inconsiderable—and that they had accumulated, in proportion to the sums assured, a large surplus before they ventured upon the slightest diminution of it.——In the commencement of a Society the probabilities of life, when compared with those in the tables of observations, will always be found much higher than in the more advanced state of it; for at this early period the lives having been just selected on account of their healthiness, while in the tables those of every description are included, it will happen of course that fewer die at first than the tables suppose. But this difference in favour of the Society must not be expected to continue always in the same degree, but rather to be perpetually lessening as the lives become older, and in consequence enjoy more nearly the same state of health as mankind in general at their respective ages.—This has been particularly the case in the Equitable Society, where it is found from long experience, that the probabilities approach nearer to those in the table;

as the lives become more advanced, till in old age they are reduced almost to an equality.—Should any Society, therefore, in the first years of its existence, presuming on the higher probabilities of life among its members, be led to reduce its premiums or in an undue degree to divide its profits; exclusive of the folly of adopting any such measures in this early period, it will, by raising false hopes and alienating that part of its capital which ought to have been suffered to accumulate for many years, be laying the foundation of future disappointment and ruin.

The motives which lead to the establishment of these companies arising generally from private interest rather than from any desire of benefiting the public, the hope of increasing the present value of their property will always be in danger of inciting them to a premature division of their capital; especially when encouraged by this fallacious appearance of success during the earlier part of their existence.—When Dr. *Price* first wrote on the subject of Life-assurances, it was in general so little understood, that the application of the Equitable Society about that time for a Charter, was rejected by our Law-officers, on the ground that their premiums were insufficient, although they were nearly twice as high as they are at present.—It is to this work, and to the great success of the Equitable Society, in consequence of the advice and assistance of the Author at its commencement, that the public are indebted for *their information, and (provided the different companies lately established should be* conducted with skill and discretion) for the important benefits arising from the more extended practice of Life-assurance.

Although this work, when first published, was not intended to be a complete treatise on the Doctrine of Annuities and Reversions, it contains, however, amidst a vast variety of other important matter, the solutions of most of the cases in which one or two lives are concerned; which are rendered the most valuable from their application to the different Societies which had then been established for the benefit of Age and Widows.—The many tables also which it contains of the probabilities of human life, deduced from real observations, form an invaluable addition to this work; for by their assistance accurate solutions are now given instead of those approximations founded on an erroneous hypothesis, which in many cases were so incorrect as to be altogether unfit for use.—To Dr. Halley, Mr. De Moivre, and Mr.

Thomas Simpson, and particularly to the latter, we owe the first rudiments and improvements of this science. But for the more accurate knowledge of it we are indebted to Dr. Price; and though he has not extended his investigations to the more complicated cases of three lives, he has laid a sure foundation for them by those tables which he has constructed, and which in the earlier and later stages of life had been ill-supplied by the hypothesis of Mr. De Moivre. I have long intended to avail myself of those tables, and to complete what Dr. Price unfortunately has left undone; nor have I yet given up my intention, but hope, should my life be prolonged and my health permit, to accomplish an object, which appears to me to be not only interesting, but to this country useful and important in the highest degree.*

The enumerations, which, within the last 12 years, have been twice undertaken of the inhabitants of Great Britain, appear to have made their number much greater than Dr. Price had supposed. They prove, however, in the most decisive manner what he has laboured to maintain, from so many documents, in opposition to most of his opponents, that the proportion of inhabitants to the number of houses is less than six to one; and, (had the surveyors of windows made correct returns of those houses which were exempted on account of poverty,) that he had even over-rated the population of the kingdom.—Had he lived, however, to have witnessed the results of those enumerations, he would certainly, as a true friend to his country, have felt pleasure in finding the number of its inhabitants so greatly to have exceeded his computations; but it must have been a pleasure mingled with pain and sorrow, when he discovered the source of his error, and found that it consisted only in relying too much on the inaccurate returns of the surveyors, and, in consequence, estimating the *paupers* of the kingdom at less than half their real number.

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* I have already given correct solutions of most, if not all of the cases in which two and three lives are involved in the computation; which have been published in the Philosophical Transactions, and also inserted in this Work. But they are given in an *Algebraical* form, and by no means fitted for an elementary, or popular treatise on the subject.

On the subject of the *National Debt*, which so much engaged the attention and alarmed the fears of the Author, I have not thought it necessary to make any additional observations. While the present expenditure continues, and the debt is every year increasing so tremendously, it can be of little use to ascertain its amount at this particular period, or to oppose any arguments to the folly and extravagance with which it is accumulating. When it amounted to 240 millions, Dr. Price considered the state of the nation as hopeless. It now exceeds three times that sum, without the prospect of its being stayed in its progress, and has certainly so far proved his fears to have been premature, or rather his ideas of fictitious credit to have been much too limited.

In proposing and recommending the establishment of a Sinking Fund, Dr. Price conceived it to be as impossible that it should become the instrument of *increasing* the debt, as that a stoppage of payment at the Bank should have produced that effect.—Had he, however, survived a few years, he would have had melancholy proofs of the erroneousness of his opinion.—He would have beheld the nation involved still deeper in debt, not only by means of the fund which was formed for its extinction, but by the very catastrophe, which he had so often deprecated as the death and destruction of all public credit.—The resources of the country have indeed proved to be far greater than Dr. Price, or even his most zealous opposers had imagined, and consequently have protracted that ruin which he anticipated. But they are not *inexhaustible; and if the same profusion and* disregard to all economy in their management and expenditure be continued, the evils which he predicted must ensue; with this only difference, that the ruin, by being longer delayed, will become more dreadful, and add millions to the number of those who are involved in distress and misery.

W. MORGAN.

London, Sept. 25, 1812.

PREFACE

TO THE

FIRST EDITION

BEFORE the Reader enters upon this Work, it will not be improper to give him the following information concerning it.

A few years ago, many gentlemen, of the first eminence in the law, formed themselves into a *Society*, for providing annuities for the widows of all such persons in judicial offices, barristers, civilians, and solicitors, as should chuse to become members. A plan was agreed upon and printed, but, some doubts happening to arise with respect to it, the directors resolved to ask the opinion and advice of three gentlemen, well known for their skill in calculation. This occasioned a further reference to me; and the issue was, that the plan being found to be insufficient, the whole design was laid aside.

About the same time, several other societies were formed with the same views; but all on plans alike improper and insufficient. Finding, therefore, that the public wanted information on this subject, I was led to undertake this work; imagining, that it might be soon finished, and that all I could say might be brought into a very narrow compass. But in this I have been much mistaken. A design, which I at first thought would give little trouble, has carried me far into a very wide field of inquiry: and engaged me in many calculations that have taken up much time and labour. I shall, however, be sufficiently rewarded for my labour, should it prove the means of preventing any part of that distress, which is likely to be hereafter produced by the societies now subsisting for the benefit of widows.——I have proved the inadequateness of their plans, by undeniable facts and mathematical demonstration.—I have, further, given an account of some of the best plans, which are consistent with a sufficient probability of permanency and success.——Should, therefore, any of these societies determine to reform themselves; or should any institutions of the same kind be hereafter established, they will here find direction and assistance.

In Question VI. Chap. I. a general method is described of finding the values, in *single* and *annual* payments, of all life-annuities which are to begin after a given term of years; and, in the 4th Section of the 2d Chapter, the plans of the societies for granting such annuities are particularly considered, and proved to be extremely deficient. ——Indeed, the general disposition which has lately shewn itself to encourage these societies, is a matter of the most serious concern; and ought, I think, to be taken under the notice of the Legislature. The leading persons among the *present* members will be the *first* annuitants; and they are sure of being gainers: And the more insufficient the scheme is, on which a society is formed, the greater will be the gains of the first annuitants. The same principle, therefore, that has produced and kept up other *bubbles*, has a tendency to preserve and promote these; and, for this reason, it is to be feared, that, in the present case, no arguments will be attended with any effect. The consideration, that "the gain made by some in these societies, "will be so much plunder taken from others," ought immediately to engage all to withdraw from them, who have any regard to justice and humanity; but experience proves, that this argument, when opposed to private interest, is apt to be too feeble in its influence.

It cannot be said with precision, how long these societies may continue their payments to annuitants, after beginning them. A continued increase, and a great proportion of young members, may support them for a longer time than I can foresee. But the longer they are supported by such means, the more mischief they must occasion.—So, a tradesman, who sells cheaper than he buys, may be kept up many years by increasing business and credit; but he will be all the while *accumulating* distress; and the longer he goes on, the more extensive ruin he will produce at last.

In the latter end of the first Chapter, I have stated very particularly, the method of computing the values of *assurances* on lives and survivorships, in all cases where no more than two lives are concerned: And, in the third Chapter, I have pointed out a considerable error, into which there is a danger of falling in computing some of these values. The societies and offices for transacting business in this way, are very useful; and it is necessary that they should go upon the best principles, and possess all the information that can be given them.

But there is no part of this work in which the public is so much concerned, as the sixth Chapter. It will be there proved, that had the sums raised for public services since the REVOLUTION, been much greater than they have been, the increase of the public debts to their present state might have been prevented in the easiest manner, and at a trifling expence. A method, likewise, of reducing within due bounds these debts, heavy as they now are, will be proposed.—All competent judges will, I believe, see, that this method being founded on the most perfect improvement that can be made of money, is the most expeditious and effectual that the natures of things admit of. Nor, in my opinion, if the nation is not yet too near the *limit* of its resources, can there be any good reason against carrying it into execution.——It is well known, to what prodigious sums, money, improved for some time at *compound interest*, will increase.[a] A state, if there is no misapplication of money, must necessarily make this improvement of any savings, which can be applied to the payment of its debts. It need never, therefore, be under any difficulties; for, with the *smallest* savings, it may, in as little time as its interest can require, pay off the *largest* debts.

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[a] A penny, so improved from our Saviour's birth, as to double itself

In the *first* Essay I have made many observations on the expectations of lives, the pernicious influence of great towns on health, and manners, and population; the increase of mankind; and other subjects in the doctrine of Annuities and Political Arithmetic.——In the second Essay I have stated carefully the proper method of forming Tables of the probabilities of human life, from given observations: And, at the close of this work, besides several new Tables, I have thought it necessary to give Mr. *Simpson's* Tables of the values and expectations of LONDON lives.——I have

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every 14 years, or which is nearly the same, put out to 5 *per cent.* compound interest at our Saviour's birth, would, by this time, (that is, in 1773 years) have increased to more money than would be contained in 150 millions of globes, each equal to the earth in magnitude, and all solid gold.*

* This computation is made on the supposition that a cubic inch of gold is worth about 35*l.*—But since a cubic inch of water is known to weigh 254 grains nearly, and the specifick gravities of *standard* gold and water are to each other as 18,888 to one, it will follow that a cubic inch of gold, according to the *Mint* price of 3*l.* 17*s.* 10½*d.* per oz. will amount to 38*l.* 10*s.* 1¼*d.* and consequently that a penny improved at compound interest so as to double itself once in 14 years, from the time of our Saviour's birth to the year 1773, would have accumulated only to 107 millions of such globes as are mentioned by Dr. *Price*,—but this is abundantly sufficient to prove the strength of his argument.—In the year 1792, it would have accumulated on the above supposition to 274 millions of those globes.

If instead of supposing money to double itself every 14 years it be improved at 5 *per cent.* compound interest, the accumulation in 1791 years will amount only to 14[illegible] millions, and in 1773 years to no more than 60 millions of globes. The difference therefore between the supposition of money's doubling itself every 14 years and of its being improved at 5 *per cent.* compound interest, is far from being inconsiderable, especially in a long term of years. M.

also, in the notes at the end of the first Volume, given the Demonstrations of the Answers to the *Questions* in Chap. I. *These Demonstrations I have chosen to keep out of sight in the body of the work, in order to avoid discouraging* such readers as may be unacquainted with mathematics.

Upon the whole. A great part of the work now offered to the Public is, I believe, new; and I am in hopes also, that it will be found to contain some improvements in those branches of philosophical enquiry, which are the subjects of it.

PREFACE

TO THE

THIRD EDITION.

THAT favourable reception of this Work, which has occasioned the present Edition of it, so soon after two former editions, is such a proof that it has been of some use to the public, as amply rewards me for the attention and labour which I have bestowed upon it. In revising it on the present occasion, I have been anxious about improving it as far as possible. Several additional facts and observations have been inserted in different places, particularly in the First Essay and the Postscript to it.—That part of the Second Section, Chap. II. which treats of the *Scotch* establishment, has been new composed, and carefully accommodated to the more accurate information concerning it, with which I have been favoured.

The SUPPLEMENT is an addition which was made to the *second* edition.—The observations in it on the present state of our population have been a good deal enlarged.—This is a very serious and important subject. If, indeed, there has been that diminution of our people which the evidence I have produced seems to shew, it must alarm every one who wishes well to his country, and it ought to engage the immediate and vigorous attention of government.——Many differ from me in this point; and I wish I could find sufficient reason to believe as they do. Several great manufacturing towns have, I know, increased; but these are nothing to the whole kingdom; and even by their increase, our population may, on the whole, have lost more than it has gained.—In truth; it would have been strange if our numbers had not been declining; for I can scarcely think of any great cause of depopulation, which has not for the last 80 years been operating among us.

The prodigious traffic now carried on in Life-annuities, and the rage for forming and encouraging Annuity Schemes, which has for some time been spreading through the kingdom, has rendered the information which I have meant to convey in the following work particularly necessary. And I have had the pleasure to observe that it has been attended to. Several of the Annuity Societies in LONDON have been dissolved; and there is reason to hope, that those which still remain will not be able much longer to support themselves on their present plans, in opposition to the evidence of demonstration, and the calls of justice and humanity.——These *Bubbles*, however, are of little consequence, compared with that GRAND NATIONAL EVIL, which is the subject of the sixth chapter of this treatise. This is an evil on which I could not imagine, that any such efforts as mine would make any great impression. Perhaps, indeed, the united efforts of all the independent part of the kingdom would now be too weak to save us from the distress with which it threatens us.

Much has been said for some time of a plan mentioned in PARLIAMENT, at the end of the last session, for paying off the NATIONAL DEBT. This raised some expectations; and I will beg leave here to give a brief account of it.

After providing for all the current services, there remains this year (1773) a *saving* or *overplus* of £1,200,000. With this sum, and a profit of £150,000 from a Lottery consisting of 60,000 tickets, a MILLION AND A HALF of the 3 *per cent.* annuities, purchased at 90, will be paid off.—When this was proposed to the House of Commons, it was at the same time announced, that it would be the COMMENCEMENT OF A PLAN FOR PAYING OFF THE NATIONAL DEBT; for, if no extraordinary services should call for any other application of the public surplusses, the same payment, increased by the interest of former payments, is intended to be made every year while the peace lasts: And thus, reckoning compound interest at 3 per cent. SEVENTEEN MILLIONS will be paid off during a peace of ten years.

On this plan I will take the liberty, with all the deference which becomes me to the station and abilities of the proposer of it, to offer the following remarks.

1st. It implies, that there is to be a *Lottery* every year during the whole continuance of peace.—Formerly, lotteries were expedients for procuring money on more advantageous terms, to which government had recourse, when pressed by the necessities of war. They are now, it seems, to be established as *permanent* resources never to be given up or suspended.—This must shock every person who is duly acquainted with the mischief occasioned by lotteries, particularly among the lower classes of people. The rage for gaming threatens the ruin of all that is virtuous and manly among us. It is increasing fast, and wants not to be fostered by government.

2dly. The *surplus* of the present year is in part the effect of some *extraordinary* savings in the last year, (1772) which cannot be expected another year: And, I believe, that those who are best acquainted with this subject, must be sensible that there is no sufficient reason to expect, while the augmentation of the navy is continued, a constant *surplus* of so much as a MILLION *per ann.* I mean this on the supposition, that the produce of the *Sinking Fund* will continue what it is taken for this year, and what it has been the last three years, or £2,600,000. But this is certainly more than can be depended on. The difficulties of the *East India Company*; that stagnation of credit which has lately distressed the public; and many other causes, may possibly occasion *Deficiencies*. Should there, however, be an *increase*, it will be owing, I am afraid, to a very bad cause: I mean, to an increase of our importations proceeding from luxury, and turning the balance of trade against us; and, consequently, draining the kingdom of its *specie*, and leaving it more and more to the precarious and dangerous support of paper money. But,

3dly, Let the *surplus* of the public revenue prove what it will, there is too much probability that, even during the continuance of peace, some emergencies or other will be often furnishing reasons or pretences for employing it *in other ways than the payment of the* public debts. This has been the case hitherto; and from the year 1730 to the present time, it has never happened, that we have gone on above three or four years together employing *surplusses* in discharging debts. Though in profound peace, there have been calls for a different application of them; nor can I imagine what reason there is for believing that our circumstances are so much changed for the better, that there will arise no such calls for ten years to come, should the peace last so long. But,

4thly, The most capital defect in this plan is, that its operation is to cease as soon as a war begins. That is; it is to cease at the very time when it would operate to most advantage, and make the quickest progress in redeeming the public debts. This has been demonstrated in the chapter on public credit in this Treatise, and in my *Appeal to the Public on the Subject of the National Debt.*

Is it then any wonder, that such a plan has had no effect on public credit?—Does it mean any more than that the surplusses of the revenue shall be applied to the discharge of our debts, when there are no other uses for them?—And was there ever a time when this was not done? Is not this the very plan we have been pursuing these forty years, and to which we owe our present incumbrances? Certain it is, that nothing but a plan that shall go on operating uniformly in *war* as well as in *peace*, or the establishment of a permanent fund that shall never be diverted; that is, in other words, a return to the scheme adopted by the legislature in 1716; and which even now stands established by law, but which, through the unpardonable misconduct of men in power, has been defeated of its good effects? Nothing, I say, but this can do us any essential service; or, in our present circumstances, be much more than trifling with the difficulties and dangers of the public.—Establish such a fund—Consign it to a particular commission, acting under penalties, in such a manner as shall take it out of the hands of the *Treasury*, and form a check even on the *House of Commons* itself.—Supply from time to time all deficiencies just as if no such fund existed; and, by these and other measures, convince *the kingdom that something effectual is* meant, and that the public debts are indeed in the way to be extinguished.—LET THIS BE DONE; and we may soon see a new state of things; public credit may revive; and the kingdom enjoy at least a chance for being preserved.—By the confidence which such a measure would give in government security; but more especially, by the increasing sums which would be thrown annually into the public markets, and returned to the public creditors, the 3 *per cents.* would soon be raised to *par*, and in some time probably far above *par*. It is well known, what an effect *borrowing* every year has in sinking funds. Paying every year would certainly have an equal contrary effect. In a time of war, particularly, it would give such a demonstration to the public, that an *irrevocable* plan of redemption was at last established, as could not but produce the happiest effects. It would indeed in these circumstances be necessary to borrow an extraordinary sum annually equal to the appropriation. That is; supposing the fund to set out with a million *per ann.* it would be necessary to borrow so much more annually than would have been wanted had the fund been capable of being diverted. But this being done to convey a conviction with which the very power of borrowing was connected; and to preserve a *fund* on which the very being of the state depended; no bad consequences could follow. The annual charge on the public, occasioned by the war, would be even *less* than it must have otherwise been. For, let us suppose *ten millions* necessary to be borrowed every year to defray the expences of war, *nine millions* only of which would have been wanted, had not the million *surplus* been locked up.—*Suppose farther, that the scheme, by keeping up public* credit, and throwing money every year into the hands of *lenders, enables government to borrow at £1 per cent.* less interest than would be otherwise required; that is, at 4 instead of 5 *per cent.*—In these circumstances, there would arise a present saving to the kingdom of £50,000 *per ann.*; for the interest of *ten millions* at 4 *per cent.* is £50,000 less than the interest of *nine millions* at 5 *per cent.* And such a saving, repeated every year of a war, would be an object of some importance to the kingdom.—Indeed, there may be no possibility of conceiving what important effects in this way, the establishment of such a scheme might produce. During its progress in discharging our debts, and before it could give any relief by the annihilation of taxes, it might *save* the kingdom, by preserving it from difficulties which would have sunk it. And every one must be sensible of this, who has considered what danger there is that a war, should it become unavoidable before our debts are put into any certain course of redemption, will either entirely overwhelm public credit, or so much weaken it, as to produce an impossibility of borrowing except on very exorbitant interest, and, consequently, of finding taxes sufficiently productive to pay such interest. The general apprehension now is, that the nation is overloaded; and that its debts will never be paid. This keeps the funds near 18 *per cent.* lower than they were in the last *peace*. In the next war such apprehensions will increase, and produce great danger. But should it be then seen, that a plan for redeeming our debts, the most efficacious possible, was going on; and, in consequence of being guarded in some such manner as I have hinted, *would* not (or *could* not *easily*) be revoked; in these circumstances, all danger would be so far lessened, that it might be practicable to find new taxes which would support the expences of war during the operations of the scheme.

But I am got far beyond the limits I prescribed myself when I begun this Preface.—*As the national debt is a subject unspeakably interesting to this nation, I could not allow myself to omit any thing that appeared to me of consequence upon it; and the Reader of this Treatise will on this account, I hope, excuse me, if I have detained him here too long and too unprofitably.* Much has been before said on this subject by writers of more consequence to no purpose; and we shall pursue the path we are in, till the edge of the precipice towards which we are advancing awakens us, and ruin becomes unavoidable.—The distress occasioned by the shock lately given to the bubble of paper-credit, is, I am afraid, a prelude to greater calamities, and a warning to prepare for them.

PREFACE

TO THE

FOURTH EDITION.

THIS work having been for some time out of print, I resolved about four years ago to prepare for the press a new edition of it, expecting that I should have only a few corrections and additions to make of no particular consequence. But in this expectation I have found myself greatly mistaken. Such a variety of new matter came in my way, and such means of improving this work were communicated to me, as have led me to bestow upon it more attention and labour than can be easily imagined, and to increase it from one to two volumes.

It is probable that nothing could have engaged me to undertake so much labour had I foreseen it; but having begun, I could not avoid going on; and I was encouraged by the reflection on the favourable manner in which the former editions of this work had been received, and also by the hope that on *one* subject of human enquiry I should be able to produce a work more complete than any that the public has been yet furnished with.

The additions of most consequence in the present edition are the following.

There has been added to the *Second* Chapter an account of several *foreign* Societies; and a continuation of the history of such annuity societies as are still subsisting in *London*, to the time when that Chapter was printed off; that is, to the beginning of the year 1782.——The largest of *these additions respect the Amicable Corporation for Perpetual Assurances* at *Serjeant's Inn*; and the *Society in Chatham Place* for *Equitable Assurances on Lives and Survivorships.*——The former of these Societies, should they think proper to look into these observations, will, I am persuaded, find that they require their attention.——But it is the Society last mentioned, which, through the whole of this Treatise, I have had chiefly in view. Having for many years been concerned in advising this Society (the first of the kind in the world; and increasing fast), I have been anxious about giving it all the information and assistance possible.——The additional observations addressed to it in this edition will be found chiefly in this Volume from page 181 to 191; and at the end of the Remarks on the 36th Table in the 2d Volume. And, from these observations, it may be learnt in particular, that though this Society, in consequence of having happily begun too high, has already found itself capable of making great abatements in its demands, it is still capable of making farther abatements.

In the former editions of this work, I had intimated that a publication of the tables by which its business is transacted would be proper, together with an account of the principles assumed, and the method taken in composing them. This is an information to which the public has a right, and which is now given it in different parts of this work, and particularly in the Second Volume at the end of the Remarks on the 36th Table.

Many corrections have been made and several additional notes inserted in the First Essay. Some of these have been occasioned by two late publications; one by Mr. *Wales*, master of the mathematical school at *Christ-Church*; and the other by Mr. *Howlet*. See, particularly, the notes in p. 17, 24, 26, and 27, Vol. II.——The Table in p. 70, Vol. II, is also now first inserted.

To the fourth Chapter a *Postscript* has been added, the *principal intention of which is to point out the reasons for discarding the valuation of single and joint lives derived from Mr. De Moivre's hypothesis*; and also to describe a method of computing these values from any given tables of mortality, which, while it leaves no possibility of any mistakes, renders such computations as easy and expeditious as their nature will allow.

The third Essay[a] is the same that it has been in all the former editions. But to the *Fourth*[b] some additions have been made; particularly, the notes in p. 88, 90, and 91, Vol. II. The first two of these notes have in view a remark of Mr. *Wales's*; and the design of the last is, to retract an assertion in the former editions concerning the duration of life in old age in great towns, and to shew the reason why a greater proportion of the inhabitants of *London* died formerly in old age than have died lately.

But the additions of most consequence are the Tables in the Second Volume, with the remarks explaining the construction, use, and application of them.——In the general introduction to these Tables, and in the remarks on Table XVI. p. 306, it is shewn (and, I think, undeniably) that the Tables of the values of lives deduced from the *London* bills of mortality err only by giving them too high; and that, with respect to the main body of the inhabitants, the unfavourableness of *London* to the duration of life continues much the same that it used to be.

Tables I. II. III. and IV. are an abridgement of Mr. *Smart's* tables of compound interest, and contain all that is important in them.

The XVIIth Table, shewing the mean probabilities of the duration of life according to a register of mortality at NORTHAMPTON, has been inserted in all the former editions; but it is now given more correctly; and tables deduced from it have been added, of the *expectations* of Life and the values of *single* lives and of any *two joint* lives at all ages, and for three rates of interest. The labour of computing these tables was undertaken in order to set aside all occasion for using the defective valuation of lives founded on Mr. *De Moivre's* hypothesis; but not having been able to finish these computations till a great part of this Treatise had been printed off, I have been obliged to continue the use of the old tables so far as to take from them many of the examples of the solutions of questions in the first and following chapters.

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[a] In the present edition, the third *Chapter*.——M.

[b] Now the *second* Essay in the *second Volume*.——M.

When tables of the values of *two joint* lives are given, the values of *three joint* lives may be deduced from them, with perfect ease, by Mr. *Simpson's* rule inserted in the Second Volume of this Treatise.[c]

This rule saves so much trouble, that I have thought it worth while to procure calculations of the XXXVth and XXXVIth Tables, on purpose to determine how far it may be depended on. The result is given in the remarks and comparisons at the end of those Tables. And it seems to appear that it finds the values of *three joint* lives so nearly as to leave little room for wishing in this instance any greater degree of correctness.

The rules from p. 221 to p. 233, Vol. I. describe a method of deducing, with sufficient accuracy, the value of any life or number of lives at *all* rates of interest, from the correct value given at *one* rate of interest.——In computing, therefore, tables of the values of lives, according to any given observations, no more will hereafter be necessary than to compute them for *any one* rate of interest.

All improvements, however, of this kind would be of little consequence, were there no tables which state correctly the laws that govern human mortality in different situations. One principal part of my business in this work has been to frame such tables; and any one who will look over the collection of tables in the Second Volume, and particularly those from the 41st to the 47th Table, will see that I have been furnished with the best means of doing this.

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[c] See Remarks at the end of the 36th Table. M.

With respect to the Tables, in particular, deduced from the *Swedish* Observations, I cannot hesitate to pronounce that they exceed in correctness every thing of this kind which has been hitherto offered to the public; and that *nothing is wanting to make our knowledge in this instance complete, but similar observations in other kingdoms.*—By these Tables I have been enabled to state minutely the different rates of mortality at all ages among males and females; and to form tables of the values of single and joint lives for *each* sex, as well as for both sexes collectively; in consequence of which, I have been farther enabled to determine the increase of the values of annuities payable during survivorship, occasioned by the longer duration of lives among females; and thus to furnish a direction of some importance to the various societies in this kingdom and abroad for providing annuities for widows.

I must not in this place neglect to acknowledge the great obligation I am under to Mr. WARGENTIN of the Royal Academy of Sciences at STOCKHOLM; for the communications which have enabled me to make the additions to this Treatise last mentioned. It will be found also, that I have been much indebted to Mr. OEDER of *Oldenburgh* for several useful communications.

In the observations and rules in page 200, &c. Vol. I. I have given a general account of the method of computing the values of *presentations* to livings, and of the renewals of leases held either for terms certain or for lives.

The rules from p. 263 to p. 271, Vol. I. shew how to deduce, from the values given of any annuities payable *yearly*, their values when payable *half-yearly* or *quarterly*, or when secured by land and payable half-yearly. And the last Table in p. 271, exhibits the particular differences between these values for two rates of interest.

The

The collection of Tables is preceded by a SUPPLEMENT which formed a part of the Third Edition of this work. In the present Edition several notes have been added to this Supplement; but the addition of most consequence is the POSTSCRIPT on the subject of the population of the kingdom.——In the former Editions, and also in the publication entitled an *Essay on the Population of England from the Revolution*, &c. I gave an account of several facts which seemed to me to shew, that our population has declined. Great pains have been taken to prove this to be a mistake.[d] In the *Postscript* just mentioned, I have entered a little farther into this controversy; and it will appear that though I still retain my former opinion, yet I wish to be considered as far from being decided in it, and therefore as open to receive any evidence which can be produced to overthrow it.

Being willing to comprize in this Edition all that I have written on the duration of human life, and the values of life-annuities, I have inserted Three Essays on these subjects, which have been published in the *Philosophical Transactions of the Royal Society*.

A pretty copious Index closes the whole.

The additions I shall last mention are those which relate to public credit and the national debt; and I have chosen to mention them last, on account of their particular nature and importance.

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[d] Much has been said also about a mistake into which it is supposed I have fallen in estimating the quantity of gold coin in the kingdom. The truth, in this instance, is briefly this.—The third proclamation for calling in the gold coin brought in near *double* the sum that was expected. In consequence of this, an estimate which I had published in the First Tract on Civil Liberty, proved short of the truth about *three millions and a half*; and it appears now, that, exclusive of two millions purchased by the bank and melted into bars, the gold coin of the kingdom was (in 1773) about *sixteen millions*, instead of *twelve millions and a half*, as I had reckoned it. And this is the account I have given in the *last* edition, p. 74, of the Tract just mentioned, except that being then not informed of the coin purchased by the bank, I have not mentioned it.

In the Preface to the Third Edition, (see p. xxii) I took notice of a plan announced in 1773 by Lord *North* to the House of Commons, for paying in the ten following years 17 millions of the public debt. It is necessary I should here just mention, that this plan was never afterwards heard of.——The remarks I have made upon it, were followed with a proposal for expediting a plan of redemption in such a manner, as to cause an appropriation of a million *per ann.* to discharge, in forty years, a *hundred millions* of the public debts then bearing 3 *per cent.* interest. This proposal has not been continued in this Edition, because I intend soon to lay before the public a plan more efficient, and better adapted to the present state of our funds. I must, however, observe that, having now no hope that an efficient plan of redemption will ever be established, I think with regret of the time and attention I have bestowed on this subject. Nothing relieves me, but the reflection that the object about which I have lost my time, has been the removal of an evil which, if no such measures as I have proposed are adopted, must bring on a *catastrophe* which will make this country a warning and a terror to the world.

At the end of the chapter on public credit I have, in this Edition, inserted a brief history of the Sinking Fund; and also a particular account of the increase of the public debts from 1776 to 1783.[e] and of the state of our finances at the time of signing the Preliminaries of peace in *January* last.[f] This account is, I believe, as correct as it is possible at present to make it; and I have chosen for many reasons that it should form a part of this work. Hereafter, probably, it will be read with amazement. Our folly, in this instance, is without example. Lord NORTH enjoys the singular distinction of having contributed more to it than any former minister. By a war which has degraded the kingdom, and a dissipation of treasure which was never equalled, he has, in the short compass of seven years, doubled a debt before too heavy to be endured. And let future generations rise up; and, if possible, let them call him—*Blessed*.[g]

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[e] This account was omitted by Dr. *Price* in the 5th Edition, but I endeavoured to supply the deficiency in the sixth Edition by giving the amount of the public debts in 1786, and also at the time of signing the Definitive Treaty of Peace in April, 1802, (p. 337, Vol. I.)—M.

[f] These statements were omitted by the author in the fifth edition.—M.

Newington-Green, March 29, 1783.

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[g] Had Dr. Price lived to witness the profusion of the last ten years,* he would probably have moderated the severity of this censure—The debt which appalled him, when it amounted to 230 millions, has lately swollen to the enormous mass of 540 millions,† and the yearly expenditure of a peace-establishment which he considered as insupportable at 16 millions, will in future require an annual taxation of more than 34 millions!—The paper-circulation which he so justly deplored when he first published this work, appears to have been then only in its infancy; it has now completely inundated the country. The coin has disappeared, and the Bank, for whose credit he was so apprehensive, has for many years ceased to pay its notes in specie. The American war, which he regarded as so injurious and disgraceful to the kingdom, has been succeeded by another infinitely more ruinous, and more degrading in its consequences. Had he therefore lived to witness these, and a long train of other calamities which are too deeply felt to need the recital of them—had he lived to behold the Man, during whose administration they were produced, retiring from office in triumph, and congratulating the nation on the envied state of prosperity to which it had been lately exalted, he would have changed the object of his resentment, and acknowledged that Lord *North* had long lost the distinction assigned to him in this preface.—M.

* This was written in 1803.

† See page 338 in this Volume.

PREFACE

PREFACE

TO THE

SIXTH EDITION.

THE following invaluable work is one of the fruits of a life which was uniformly devoted to studious enquiry, and to the promotion of the best interests of mankind. The motives that first led the author to engage in these labours were the most humane and honourable, and he considered the success that attended them as the highest reward and gratification he could enjoy. Even in preparing his notes for the former edition, which unfortunately were never completed, he mentions his life as then drawing near its close, but that he had reason to reflect with satisfaction on the time which he had employed in those pursuits.

The different Prefaces which have been affixed to the several editions published by the Author himself, render it unnecessary to enter minutely into the origin and progress of this work. But it may not be improper to observe, that the advice and instruction which it contained could not possibly have been communicated at a more seasonable opportunity, than at the time of its first publication in 1769. —The various Societies for the benefit of age and widows which were then continually rising up to allure and to defeat the hopes of the ignorant and distressed, were become an object of serious concern, and if the evil had not been effectually opposed, it would certainly have proceeded to a much more alarming extent. On the first appearance, however, of this work, the rage for establishing *new* Societies

cieties immediately subsided, a *partial* reformation took place in some of those which had been already formed, and in a short time the greater part of them, convinced of their mistakes, dissolved themselves. A few, indeed, persevered in an obstinate adherence to their original plans, but they have long since exhibited a melancholy proof of their own folly, and of the truth and justice of the admonitions which had been wasted upon them.

It was Dr. *Price's* intention to have written a new Preface to the former edition, in which he meant to have continued his account of the state and progress of the few Societies which then remained, and also to have inserted whatever additions and remarks he might have thought necessary to the improvement of the following volumes.—But he died soon after the *first* of them was printed, and left only a few detached hints and observations in regard to the plan which he was to have adopted.—Being anxious to exert every effort in my power towards fulfilling his intentions, I then endeavoured, though in a much inferior manner, to pursue the method which he had pointed out for himself; but in the present edition I have ventured to deviate from that plan, by introducing the remarks which I had formerly inserted in the preface, together with some additional observations, in the form of notes to the different chapters which treat of those Societies.

Besides the peculiar satisfaction which he derived from this work as having been instrumental in doing much good, Dr. *Price* was also accustomed to reflect on it with pleasure, as it contained, in his opinion, many improvements in the doctrine of Annuities and in Political Arithmetic.—It was his wish therefore to have rendered it as complete as possible; and the great number of tables and valuable observations with which he has enriched each edition, and particularly the *fourth*, are a proof of his zeal and success in accomplishing this purpose.—To the fifth edition a new table was added of the values of two joint lives, computed under his direction from the probabilities of life at *Northampton*, reckoning interest of money at 6 *per cent.* and also three other tables of the values of single lives, communicated and computed by myself, from the same table of observations at the several rates of 6, 7, and 8 *per cent.* These I believe were all the *new* tables, excepting those in the Appendix, which Dr. *Price* meant to have added to that edition :—nor did he seem to entertain the most remote idea of making any material alterations either in the arrangement or the matter of the second volume, which he did not live to correct.—When that edition was put to the press it was done in some haste, and his other engagements prevented him from attending to the correction of it before the first two chapters were printed off. Had not this been the case, he meant instead of the present answers to the 11th and 12th questions in the first Chapter, to have substituted others from a paper communicated by myself to the *Royal Society* in the year 1788, which was honoured with their approbation, and published in the 78th Volume of their Transactions. But being disappointed in this intention, he reserved what he had to say on the subject for a note at the end of the second volume.—As far as relates to these questions I have endeavoured to supply the deficiency. I have also explained some of the notes at the end of the first volume which appeared to me to be too concise, and have deduced from the real probabilities of life other solutions of such Problems as had been derived from Mr. *De Moivre's* hypothesis,—intending by this means to give a further proof that it can seldom or ever be necessary to have recourse to such an expedient in the doctrine of Annuities.—In consequence of a reference made by Dr. *Price* in a note at the end of the third Chapter, (Vol. I.) I have inserted all the different rules respecting the values of reversions depending on survivorships between three lives, which I had communicated to the *Royal Society* in the years 1789, 1791, 1794, and 1800, and which were published in the 79th 81st, 84th, and 90th Volumes of the Philosophical Transactions,

actions. These, like all the other additions I have made, have been kept separate from the work itself; so that, if any errors have been committed, they may be easily ascribed to the real author.

As all the solutions in the first Chapter, except those of the 11th and 12th questions, are strictly true[a] according to any table of the probabilities of life, I do not know that this part of the work admitted of any other improvement than the addition in those two questions of correct rules in lieu of the approximations which had been taken from Mr. *Simpson's* Select Exercises. But in regard to the second Chapter, which contains an account of the different Societies for the benefit of age and widows, the circumstances and even plans of some of these have undergone such considerable alterations since the year 1773, when the fourth edition was published, that it became necessary, in this and the former edition, agreeable to the author's own intention, to give a more accurate account of their state.

In the first Volume of this work[b] an account is given of a plan for enabling the labouring poor to provide support for themselves in sickness and old age by small weekly savings from their wages. Since the first publication of this plan Dr. *Price* had bestowed great attention on the subject; and being furnished with more information than he then possessed, he was enabled to compute the valuable Tables which are inserted in the Appendix.—These were intended to have served as the foundation of a plan to be established by the legislature, which should hold forth such particular encouragement to the poor as should induce them to give it the preference to those insufficient and erroneous plans which have been so generally adopted in this kingdom. Nothing could be more important than such a design. It was founded on the best principles, and had the happiest tendency to assist the poor by encouraging industry and saving, and by raising them above the wretched necessity of depending upon the parish for subsistence when rendered incapable of providing for themselves. The bill for establishing this plan was brought into parliament about twelve years ago, and after having passed the House of Commons was rejected by the Lords.—But it is to be hoped that this want of success will not prevent the subject from being resumed hereafter: For its usefulness and importance, when properly understood, can hardly fail to engage every branch of the legislature in its support. It is observed by Dr. *Price*[c] that the poor's rate in the year 1777 amounted to £1,556,804. Since that time it has been doubled; and the truth seems to be, that there is a growing distress among the poor, occasioned most probably by the price of provisions having increased more in proportion than the price of labour; which renders it almost impossible for a poor man, if he have a wife and family, to subsist on his wages.—This melancholy truth might be proved in the most convincing manner from the valuable information communicated to Dr. *Price* by Mr. *Moreton Pitt*, the member for Dorsetshire, whose humanity has led him to make particular enquiries into the state of the poor of this Kingdom.—But it would be foreign to my purpose to enter into disquisitions of this kind, though I cannot help observing how much it is to be lamented that others have not been stimulated by this benevolent example to make similar enquiries, and to interest themselves in adopting such measures as might effectually relieve the distresses of the labouring poor. By the formation of rational plans for this purpose the many wretched schemes which have so often risen up to delude their hopes would disappear, the indigent labourer and his family might then look forward without anxiety to seasons of sickness and old age, and the more wealthy enjoy the advantage of a reduction in the enormous expence with which the parishes are loaded for the maintenance of their poor.—It is indeed a great misfortune that the different parochial clubs established in this kingdom should in general be founded on such erroneous principles, and that a disposition so laudable in itself should not be properly directed and encouraged by the legislature. The funds of these Clubs have, it is true, been preserved by a late Act of Parliament from the depredations of their treasurers. But it would have been as well, perhaps, if provision had been made in the same Act *against the formation* of such clubs or of any Society for the benefit of Age, Widows, &c. without *having previously* submitted their plans to some persons competent to judge of their efficiency.—At present they are formed from the crude suggestions of the most ignorant, who in order to procure a sufficient number of subscriptions often propose the most extravagant terms; so that it soon becomes a matter of very little consequence to the greater part of the subscribers whether they are robbed by their treasurer, or ruined by the multitude of their claimants. But of all the phantoms which are held up to entice and to deceive the public, none are more mischievous or deserve more severe reprehension than the TONTINES which have lately prevailed so much in every part of this country. By these, while the adventurer is flattered with the extravagant hope of making his fortune in a short period and at a small expence, the worst spirit of gambling and idle speculation is called forth, and all those baneful effects which are produced by a State Lottery in *London* are extended to the remotest corners of the Kingdom. Is it not then a matter of serious concern that this mischievous spirit should derive sanction and encouragement from those who ought to exert every effort in opposing it? The annual establishment of a State Lottery is now regularly enumerated among the ways and means of the Minister; but of all the wretched measures of finance which have lately been adopted, it is impossible

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[a] The *Solutions* of the 15th and 16th questions are true, and the *rules* deduced from them are sufficiently correct for any useful purpose.—See Note (II) Vol I.

[b] Note page 149.

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[c] Note page 151. Vol. I.

possible to imagine one so unprofitable to government, and at the same time so deleterious in its consequences to the public. Wretched must be the condition of that country where the morals of the people are sacrificed to the interests of the revenue, and where the public exigencies are provided for by the pernicious assistance of such expedients. While the worst passions of the human mind are thus cherished and maintained by those who ought to exert their power in suppressing them, we may indeed lament the mischievous consequences, but it will be of little avail to oppose them by the sober efforts of reason and argument.

It was Dr. *Price's* intention, as appears from some references in the former edition, to have continued his account of the public revenue and expenditure to the beginning of the year 1791.—He had also proposed to have published the three plans for paying off the national debt, which were communicated by him to Mr. *Pitt* in consequence of an application having been made to him for his opinion and advice on that subject.—One of these plans was adopted by the minister without the slightest acknowledgment, and now forms the foundation of an Act for reducing the public debt, which was established by the legislature in 1786, and contributed more than any other measure to raise the credit of the late Administration. In order as far as lay in my power to fulfil the intentions of Dr. *Price*, I have in the first Volume inserted those plans, together with such remarks as were necessary to explain them. These formed a part of a pamphlet published by myself on the subject in the year 17 3; which having been long out of print, I thought so much at least of it might be preserved in this work, as would tend to illustrate the conduct of a Minister, who, whatever may be his pretensions in other respects, does not appear to include gratitude in the list of his virtues.—At the conclusion of these remarks I have *also added* an account of the National Debt;—not indeed at the period to which Dr. *Price* intended to have brought it, for that would have given a very imperfect idea of its amount, but to the end of the year 1801; from which it will be seen how feeble the operations of the best plans must have proved against the accelerated accumulations of the last ten years.

But the chief alterations in this edition will be found in the *arrangement* of the work. All that related to the same subject I have endeavoured, as far as circumstances would permit, to bring together; so that some of the Chapters which composed the second, now form a part of the first Volume, and on the contrary some Chapters in the first are transposed into the second Volume. By this means the whole is rendered more methodical at least, and the reader, in consequence, when studying any particular subject, will not be interrupted, as in the former editions, by having to examine different parts of the work before he finds all that has been written in it on that subject.

Hence the questions concerning Reversionary payments, and the application of those questions to the establishment of Societies for the benefit of Old Age and Widows, are followed by an account of the different Societies of this kind now existing in Great-Britain. In the next Chapter is given the method of calculating the Values of Reversions depending on Survivorships, which appears to have been written chiefly with the view of correcting an error into which Mr. *De Moivre* had fallen in the solution of those Problems.—To these succeed an account of the values of the renewal of leases, and a Chapter containing the most expeditious method of computing the values of single and joint lives according to any tables of observation. In the 5th Chapter are given Theorems for finding the difference between the values of Annuities payable yearly, half-yearly, quarterly, and momently; which, together with the algebraical notes at the end of the first Volume, contain all that was formerly dispersed throughout the different parts of the work on the subject of Annuities and Reversions.

The National Debt seems to be a subject not immediately connected with any of the other matters of which these volumes are composed. The 6th Chapter, therefore, which treats of public credit, the establishment of a sinking fund, the amount of the public debt, &c. is inserted at the end of the 1st Volume, in order that the whole of what has been written on the expectations of life, the population of the kingdom, and the method of forming tables of observations, might be connected together in the 2d Volume.

The different Chapters on these subjects constitute a great part of that Volume; and the remainder, excepting an appendix at the end containing some computations and rules for the better formation of benefit clubs, is entirely occupied by the vast variety of Tables, which render this work valuable above every other publication of the same kind. It is necessary, however, to observe that in this new arrangement not the slightest variation has been made in the text, nor has any other improvement been intended by the notes and alterations in this edition, than to methodize the general plan of the work, and as far as possible to follow Dr. *Price's* practice of extending and adapting the information it contains to the circumstances of the present time.

W. MORGAN

January, 1803.

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ERRATA.

In a Note, p. 11—in ditto, p. 23—in ditto, p. 25—and in ditto, p. 32, *for* " at the end of Vol. II." *read* " at the end of this Volume.——And in the running title at the top of page 223, *for* " Ratios" *read* " Rates."

CHAPTER I.

Questions relating to Schemes for granting Reversionary *Annuities, and the Value of Assurances on Lives.*

QUESTION I.

"A SET of married men enter into a "society for securing annuities to their widows. What sum of money, in a single "present payment, ought every member to "contribute, in order to entitle his widow "to an annuity of £30 for her life, estimating interest at 4 *per cent?*"

ANSWER.

It is evident, that the value of such an expectation is different, according to the different ages of the purchasers, and the proportion of the age of the wife to that of the husband. Let us then suppose, that every person in such a society is of the same age with his wife, and that one with another all the members when they enter may be reckoned 40 years of age, as many entering above this age as below it. It has been demonstrated by Mr. *De Moivre* and *Mr. Simpson;*

that "the value of an annuity on the *joint* "*continuance* of any two lives, subtracted "from the value of an annuity on the life "in expectation," gives the true present value of an annuity on what may happen to remain of the latter of the two lives after the other.

In the present case, the value of an annuity to be enjoyed during the *joint continuance* of two lives, each [a] 40, [b] is 9.826, according to the probabilities of life in the Table of Observations formed by Dr. *Halley*, from the bills

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[a] See Table VII. Vol. II.

[b] The values of *joint* lives and reversions, as deduced from the *Breslaw* observations, are not given in any part of this work from Mr. *De Moivre's* rules in his treatise on annuities on lives. For these rules are approximations, which give results so far from the truth, as to be, not only useless, but dangerous. In the 4th Chapter a particular account of this will be given, and also of the method in which these values have been calculated.

Mr. *De Moivre* has calculated the values of *single* lives, on the supposition of an *equal decrement of life* through all its stages till the age of 86, which he considered as the utmost probable extent of life. Thus, let there be 56 persons alive at 30 years of age. It is supposed that one will die every year till, in 56 years, they will be all dead. The same will happen to 46 at 40, in 46 years. To 36 at 50, in 36 years, and so on for all other ages. The number of years which a given life wants of 86, he calls the *complement* of that life. Fifty-six, therefore, is the *complement* of 30; 46 of 40, and 36 of 50.

This hypothesis eases very much the labour of calculating the values of lives; and at most ages between 30 and 70, or 75, it is so conformable to Dr. *Halley's* Table of Observations, that I shall not, in these questions, think it necessary to distinguish between the values of single lives bills of mortality of *Breslaw* in *Silesia*. The value of a single life 40 years of age, as given by Mr. *De Moivre*, agreeably to the same Table

lives as deduced from this Table, and the same values deduced from the hypothesis.

In order to avoid putting the reader to trouble, I have given this table among other tables in the 2d Volume. And I have also given two tables which I have formed from the bills of mortality at *Northampton* and *Norwich*. These last answer more nearly to Mr. *De Moivre's* hypothesis than Dr. *Halley's* table; and the difference between the values of *single* and *joint* lives by the *hypothesis*, and the same values computed strictly from the tables, is generally less in these tables than in Dr. *Halley's*. When therefore, in the course of this work the values of *single* and *joint* lives are mentioned, as given agreeably to Dr. *Halley's* table, it must be understood, that they are taken from the 6th and 7th tables, and given in strict agreement only to the *hypothesis*; and that for this reason, they are in reality still more conformable to the *Northampton* and *Norwich* tables.

The inhabitants of *London*, as is well known, not living so long as the rest of mankind, the values of *single* and *joint* lives there are considerably less than those just mentioned. And, therefore, whenever I have had *London* lives in view, I have given particular notice of it, and taken their values from Mr. *Simpson*, who has calculated them from the *London* tables of observation. See the Tables in the 2d Volume. (*a*)

(*a*) When this work was first published no other tables of the values of annuities on single and joint lives had been computed from real observations, than those given by Mr. *Simpson* from the very inaccurate bills of mortality in *London*. Dr. *Price*, therefore, at that time was under the necessity, in exemplifying his rules, of having recourse to Mr. *De Moivre's* hypothesis, which, agreeing nearly with the *Breslaw* table in the middle stages of life, he considered as more correct than tables deduced from the ble, is 13.20[c]; and the former subtracted from the latter, leaves 3.37, or the true number of years purchase, which ought to be paid for any given annuity, to be enjoyed by a person 40 years of age, *provided* he survives another person of the same age, interest being reckoned at 4 *per cent. per annum.* The annuity, therefore, proposed in this Question being £30, the present value of it is 30 multiplied by 3.37, or £101 2s[d].

By calculating from Mr. *Simpson's* Tables[e]

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the decrements of life in a large town, where they are always found to be much higher than among mankind in general.—In the fourth edition, however, new tables were given of the values of Single and Joint lives deduced from observations at *Northampton*, *Sweden*, &c. which have rendered this hypothesis altogether unnecessary, and enabled us to obtain the exact values of Annuities and Reversions in all cases, instead of those values which in the earlier and later periods of life do not even *approximate* to the truth.—But inasmuch as Dr. *Price* chose to continue his original examples in the two editions which he published after computing the above-mentioned tables, I have not ventured to alter them, but have contented myself with giving the correct values in the notes subjoined, which will be found in general to differ but little from those in the body of the work, owing to the examples having for the most part been confined to the middle stages of life, where the hypothesis agrees nearly as well with the *Northampton*, as it does with the *Breslaw* table of observation. M.

[c] See Table VI.

[d] The annuity by the Northampton table is worth 3.377 years purchase, or £101 6*s*. M.

[e] See Tables X and XI. Vol. II.

formed from the bills of mortality of *London*, this value comes out £102.

The difference in the value of the reversion will be inconsiderable, whether the common age is taken a few years more or less than 40. Thus married men of 30 ought not, according to Dr. *Halley's* Table, to give two-fifths of a year's purchase more, for any given reversionary annuity for their wives, than married men of 50, provided they are of the same ages with their wives; and one quarter more, according to Mr. *Simpson's* Table. If the wives are younger (as is generally the case) there will indeed be a considerable difference; for the value now determined would be £120 according to the *Breslaw* Observations, supposing the two lives to be 40 and 33, or that wives are one with another seven years younger than their husbands; and £119 8*s*. according to the *London* Observations.

QUESTION II.

" Supposing such a society as that de-
" scribed in the preceding Question, to be
" limited to a certain number of members,
" and constantly kept up to that number, by
" the admission of new members as old ones
" are lost, in consequence of their own
" deaths, and the deaths of their wives:
" What is the number of annuitants which,

"in some time after its establishment, will "come to be constantly upon it?"

ANSWER.

Since every marriage produces either a widow or widower; and since all marriages taken together would produce as many widows as widowers, were every man and his wife of the same age, and the chance equal which shall die first; it is evident, that the number of widows that have ever existed in the world, would, in this case, be equal to *half* the number of marriages. And what would take place in the world, must also, on the same suppositions, take place in this society.——In other words; every *other* person in such a society leaving a widow, there must arise from it a number of widows equal to half its own number.—But this does not determine what number, all living at one and the same time, the society may expect will come to be constantly upon it. For if every widow lived no more than a year, the society would never have more annuitants upon it, than came on in a year. And on the contrary, if none ever died, the number of annuitants would go on increasing for ever.—'Tis, therefore, necessary, in order to answer the present enquiry, to determine how long the *duration* of *survivorship* between persons of equal ages will be, compared with the *duration* of *marriage*. And the truth is, that, supposing the probabilities of life to decrease uniformly.

uniformly[f], the former is equal to the latter; and consequently, that the number of *survivors*, or (which is the same supposing no second marriages) of *widows* and *widowers* alive together, which will arise from any given set of such marriages constantly kept up, will be equal to the whole number of marriages; or *half* of them (the number of widows in particular) equal to *half* the number of marriages.—Now, it appears that in most towns the decrease in the probabilities of life is in fact nearly uniform. According to the *Breslaw*, the *Norwich* and *Northampton* Tables of Observation, almost the same numbers die every year from 20 years of age to 77[g]. After this, indeed, fewer die, and the rate of decrease in the probabilities of life is retarded. But this deviation from the hypothesis is inconsiderable; and its effect, in the present case, is to render the duration of survivorship *longer* than it would otherwise be. According to the *London* Table of Observations, the numbers dying every year begin to grow less at 50 years of age; and from hence to extreme old age, there is a constant retardation in the decrease of the probabilities of life[h]. Upon the whole, therefore, it appears, in answer to the present Question, "that according to the *three* "*former Tables* of Observations, and supposing no widows to marry, the number "enquired after is *somewhat greater* than "half the number of the society; but, according to the *London Table*, a *good deal* "*greater*."

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[f] That is, supposing that out of any given number alive at any age, the same number will die every year till all are dead. See the preceding note. That on this hypothesis, the duration of survivorship is equal to the duration of marriage, when the ages are equal; or, in other words, that the *expectation* of two joint lives, the ages being equal, is the same with the *expectation* of survivorship, may be learnt from the 18th and 20th problems of Mr. *De Moivre's* treatise on annuities; and a demonstration of it, together with a particular explanation of this subject, may be found at the beginning of the first Essay to which I must beg the reader to turn, if he is at any loss about the full meaning of what is here said.

[g] See Tables V. VIII. and XVII. Vol. II.

It must be carefully remembered, that this has been determined on the supposition, that husbands and their wives are of equal ages, and that in this case it becomes an equal chance which shall die first. In reality neither of these suppositions is just. Husbands in general are older than their wives; and, in equal ages, the mortality of males has been found to be greater than the mortality of females. For both these reasons, it is much more than an equal chance that the husband will die before his wife, or that the woman shall be the survivor of a marriage, and not the man. This will increase considerably the duration of survivorship on the part of the woman, and consequently the number enquired after in this Question. The marriage of widows will also diminish this number, and the operation of these causes will be different in different situations. But it is by no means to be expected (in the situation of the societies I have in view) that the diminution from the latter cause will be considerable enough to overbalance the operation of all the other causes which have been mentioned, and reduce the number under consideration so low, as half the number of marriages[1].

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[h] The reason of this difference between the *London* and other Tables, will be given at the end of the second Essay.

SCHOLIUM.

In *London* it appears, that there is a retardation of the decrease in the probabilities of life, which renders the duration of survivorship between two lives of equal ages, considerably longer than their joint continuance. It seems worth observing, that this is the reason why, though the probabilities of life, and therefore the values of single and joint lives, are less in *London* than in other places, yet the values of reversions depending on survivorships, are in some cases greater there. It is proper to add, that this likewise is the reason why, in calculating the values of joint lives and reversions, the present value of an annuity payable yearly to the survivor of two equal lives, may come out equal to, or even greater than, the present value of a like annuity for the joint lives. As an annuity, during such survivorship, will probably not become payable for some years, and therefore the money given for it will have time to accumulate, it is manifest, that the value of it could never be equal to the value of an annuity on the joint lives, the payment of which begins immediately, were not the observation now made true.

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[1] It will be observed hereafter, that this observation has been found to be true in fact.

QUESTION III.

"Such a society as that described in the "preceding Questions being supposed; in "what time will the number of annuitants "upon it come to a *maximum*?"

ANSWER.

In order to be more clear in answering this Question, I will first suppose the society to comprehend in it from its first establishment, *all* the married persons of *all* ages in any town or country, where the number of people continue constantly the same. In this case, the whole collective body of members will be at their greatest age at the time of the establishment of the society: and the number of members, together with the number of widows left every year, will, taking one year with another, admit of no increase or diminution. The number of widows in life together, derived from any given number coming on a society every year) will increase continually, till as many die off as are added every year; that is till they come to die off as fast as possible. But they cannot die off as fast as possible, till the whole collective body of widows are at their greatest age: or, till there is among them the greatest number possible of the oldest widows; and, therefore, not till there has been time for an accession to the oldest widows, from the youngest part of the widows that come on annually.

Let us, for the sake of greater precision, divide the whole medium of widows that come on every year, into different classes according to their different ages, and suppose some to be left at 56 years of age, some at 46, some at 36, and some at 26. The widows, constantly in life together, derived from the first class, will come to their greatest age, and to a *maximum*, in 30 years, supposing, with Mr. *De Moivre*, 86 to be the utmost extent of life. The same will happen to the second class in 40 years, and to the third in 50 years[k]. But the whole body, composed of these classes, will not come to a *maximum*, till the same happens to the fourth or youngest class; that is, not till the end of 60 years. After this, the affairs of the society will become *stationary*, and the number of annuitants upon it of all ages will keep always nearly the same.

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[k] In note (A), at the end of the 2d Volume, a rule is given by which the numbers alive at the end of any particular number of years may be very easily determined.

Such is the answer to this Question, supposing a society to begin with its complete number of members, consisting of married persons of all ages, in the same proportion to one another, with the proportions in which they exist in the world.——If it begins with its complete number of members, but at the same time admits none above a particular age: If, for instance, it begins with 200 members all under 50, and afterwards limits itself to this number, and keeps it up by admitting every year, at all ages between 26 and 50, new members as old ones drop off; in this case, the period necessary to bring on the *maximum* of annuitants will be just doubled. For, in the first place, the whole collective body of members will be 60 years in getting to their greatest age, as may easily appear from what has been just said. The annual medium of widows therefore, that will come on the society will increase continually for 60 years; it being evident, that the older any set of married men are, taken one with another, the faster they will leave widows. And after this annual medium is increased to a *maximum*, 60 years more will be necessary to bring to a *maximum* the number in life together, derived from such a *fixed* annual medium constantly coming on.——If such a society is any number of years in gaining its *maximum* of members, the time necessary to bring on the *maximum* of annuitants will be still further prolonged, and will be equal to twice 60 years with that number of years added.—Most of the societies for granting annuities to widows are of this kind: and, therefore, supposing them to gain their complete number of members in ten years, and for ever afterwards to preserve it, the number of annuitants upon them will go on increasing for 130 years—It is proper, however, to be remembered, that the increase will be quicker at first, and afterwards slower; and that, within 20 or 30 years of the end of this term, it will be so slow as scarcely to be sensible, though still real.

All who will bestow due attention on this subject must see these decisions to be just; and a demonstration of them might be given, in a form more strictly mathematical, were it necessary.

QUESTION IV.

"Suppose the members of such a society "as that described in the preceding Ques-"tions, to chuse making *annual payments* "*during the continuance of marriage* in lieu "of the sum which the reversionary annuity "for their widows is worth in *present mo-*"*ney*: What ought these *annual payments* "to be, estimating interest at 4 *per cent*?"

ANSWER.

ANSWER.

• This will be easily determined, by finding what annual payments, during two joint lives of given ages, are equivalent to the value of the reversionary annuity in *present money*.—Suppose, as in Question I. the two joint lives to be each 40, and the reversionary annuity £30 *per annum*. An annual payment during the continuance of two such lives is worth, according to Dr. *Halley's* Table of Observations, 9.82[l] years purchase. The annual payment then ought to be such as being multiplied by 9.82, will produce[m] £101.1, the present value of the annuity in one payment by Question I. Divide then £101.1 by 9.82, and the *quotient*, or £10.3 will be the answer.——This is very nearly the annual payment of all the members at an average, supposing equal numbers to offer themselves for admission of every age between 30 and 50. As much as some give less, others ought to give more, according to their excess of age. Thus, the annual payment of a married person, 30 years of age, ought to be £9.39; and of a person 50 years of age, £11.33[n].——If the values of joint lives and of the reversionary annuity are taken agreeably to the *London* Table of Observations, these annual payments will be, for 30 years of age[o], £10.9—for 40, £12.6—for 50, £14.5.

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[l] See Table VII. Vol. II.

[m] Particular notice should be taken of the method of notation here used, because it will be carried through the whole of this work.——The figures on the right hand of the full-point signify the decimal parts of £1. Thus; £101.1, is 101 and the 10th of £1 or £101 and 2s.—£9.39, is £9, and 39th hundredths of £1 or £9 7s. 10d.——£11.33, is £11, and 33 hundredths of £1 or £11 6s. 7d.——In general; it should be remembered, that 2 shillings allowed for every unit in the first place of decimals, and two-pence half-penny for every unit in the second place of decimals, will give, nearly enough, the value of the decimal part of every such expression.

If either the rate or interest is supposed lower, or wives are supposed younger than their husbands, the annual payments will be increased. But there is no occasion for pointing out particularly the difference. It may be easily found in any cases by the directions now given. There is, however, one observation which ought to be here carefully attended to.—This method of calculation supposes, that the first annual payment is not to be made till the end of a year. If it is to be made *immediately*, the value of the joint lives will be increased one year's purchase; and, therefore, in order to find in this case the annual payments required the value in present money found by Quest. I. must be divided by the value of the joint lives increased by unity, and, in this way, the preceding values at 4 *per cent.* according to the *Breslaw* Observations, will be found to be £8.62—£9.35—£10.07 [p]—According to the *London* Observations, £10,—£11.2,—£12.7.

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[n] These payments by the Northampton Table are £9.19 and £11.82 respectively. M.

[o] The value of two joint lives of 30, taken from Table XII. Vol. II. is 9.6. This subtracted from the value of the life in expectation, or from 13.1, by Table XI. gives 3.5, the number of years purchase which an annuity for a life of 30 years of age, *after* another life of the same age, is worth. This remainder, multiplied by 30, gives £105, the value in a single payment, supposing the reversionary annuity to be £30. And £105 divided by 9.6, gives £10.9, the value of the same annuity in annual payments, during the joint continuance of the two lives, according to the *London* observations.—By similar operations all the other values above given have been found.

QUESTION V.

" A society may chuse to make abate-
" ments in these annual payments, and to
" require the remainder of the value of the
" reversionary annuity to be given, in fines
" or premiums, at the time of admission. It
" may, for instance, chuse to fix the annual
" payments of all the members to 5 guineas.
" What, in this case, would be the premium
" due at admission, the annuity being sup-
" posed £30 *per annum*, and interest being
" at 4 *per cent?*"

ANSWER.

From the whole present value of the annuity in one payment, subtract the value of

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[p] By the *Northampton* Table, £8.45, £9.36, and £10.5. M.

5 guineas *per annum*, during the joint lives; and the remainder will be the answer.

Supposing the joint lives both 40, the whole present value of the annuity in one payment is, according to the *Breslaw* Observations, £101.1, by Quest. I.—The value of 5 guineas *per annum*, or of £5.25 *per annum*, during two such joint lives, is £5.25, multiplied by the value of the joint lives: that is, 5.25, multiplied by 9.82, or £51.55; and this subtracted from £101.1, gives £49.55, the answer required for two lives at the age of 40. The answer found in the same way for two lives whose common age is 30, is £46.5,—and for two lives at 50, £50[q].

According to the *London* Observations, these values are, for two lives, at 30, £54.6.—At 40, £59.4.—At 50, £63.3.

If the first of the annual payments is to be made immediately, the true answer will, in every instance, be the values found in the manner now directed, diminished by the annual payment; or, in the present case, 5 guineas less than the values specified.

The values, in *single* and *annual payments*, of any other reversionary annuity, will be as much greater or less than these, as the annuity itself is greater or less.

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[q] These three values by the *Northampton* Table are £49.75, £44.65 and £53.05. M.

QUESTION VI.

"A person 35 years of age wants to buy "an annuity, for what may happen to remain of his life after 50 years of age. "What is the value of such an annuity in "*ready money*, and also in *annual payments*, "till he attains to the said age; that is, in "annual payments for 15 years, subject in "the mean time to failure, should his life "fail?"

ANSWER.

The present value of such an annuity is the *present* value of a life at 50, in money to be received 15 years hence, and the payment of which depends on the contingency of the continuance of the given life 15 years. That is; it is equal to the value of a life at 50, multiplied by the present value of £1 to be received at the end of 15 years, and also by the probability that the given life will continue so long.—A life at 50, according to Mr. *De Moivre's* valuation of lives, and reckoning interest at 4 *per cent.* is worth 11.34 years purchase. The present value of £1 to be received at the end of 15 years, is, by Table I. Vol. II. 0.5553. And the probability that a life at 35, will continue 15 years, is, according to the *Breslaw* Observations $\frac{346}{490}$[r]. And these three values, multiplied by one another, give 4.44, or the number of years purchase that ought to be given for the annuity.—The annuity then being supposed £50, its value in present money is £222.

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[r] The probability that a given life shall continue any number of years, or reach a *given age*, is (as is well known)

In order to find this value in *annual payments*, while the given life is attaining 50, it is necessary to find the value of an annuity for 15 years, subject to failure on the extinction of the given life. And the value of such annuity is, evidently, the last value subtracted from the value of the given life; or, in the present instance, £4.44, subtracted from £13.97. (See Table VI. Vol. II.) that is, £9.53.—£222 then, being the present value of an annuity of £50 for the remainder of a life now 35, after attaining 50; and 9.53 being the number of

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known) the fraction, whose *numerator* is the number of the living in any Table of Observations opposite to the *given age* and *denominator*, the number opposite to the present age of the given life. Thus, in the present instance; 346 is the number in Dr. *Halley's* Table opposite to 50, and 490 the number opposite to 35. $\frac{346}{490}$ (or the odds of 17 to 7) is, therefore, the probability that a person whose age is 35 shall attain to 50, or live 15 years. In the same manner it will appear, that, according to the same Table, the probability that a person at this age shall live 25 years, is $\frac{242}{490}$; or nearly an even chance.

At *Norwich* and *Northampton* a person at the same age, has an even chance of living 26 years; but in *London*, scarcely 20 years. See Tables V. VIII. IX. and XVII. at the beginning of Vol. II. I will add, though foreign to my present purpose, that a person at the same age has in these towns a better chance of living one year, than in *London*, in the proportion of 3 to years purchase, which ought to be given for an annual payment to last 15 years, if a life now 35 lasts so long, it follows that the value of the same annuity in annual payments, till this life attains 50, is £222 divided by 9.53; or £23.3.

This calculation supposes, that the first of the annual payments is not to be made till the end of a year. If the first payment is made immediately, the value will be, the *single payment* divided by the value of the life for the given term increased by unity; that is, in the present case, £222 divided by 10.53; or £21.08[s].

If the value of the annuity is required in a single payment, over and above any given annual payment; deduct the value of the annual payment from the whole value in a single present payment, and the remainder will be the answer.—Thus, let 5 guineas, in the present instance, be the given annual payment for the assigned term; and let the enquiry be, how much more in present money the supposed annuity is worth. By what has been just said, 9.53, multiplied by 5 guineas, that is, £50 is the value of the annual payment; and this sum deducted from £222, leaves £172 the answer.

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[s] It never happens that the annual payments are made in this manner—The usual, indeed the invariable method is, to advance the first payment immediately, and the remaining ones at the *beginning* of each of the following years; so that the number of payments shall be equal to the number of years.—The single payment therefore ought to be divided by the value of the life *for one year less than the given term* increased by unity; hence £222 in the present case, instead of being divided by £10.53, should be divided by £10.18, (or the value of the life for 14 years, increased by unity,) and consequently the annual payment will be £21.8, which agrees nearly with the same payment found by the *Northampton* Table. M.

If the annual payment begins immediately, its value is 10.53, multiplied by 5 guineas, and the answer comes out £166 75[t].

In this way may be found the value, in single and annual payments, of any other annuity, payable to an assigned life, after a given term of years, taking any valuation of lives or interest of money. But care must be taken to remember, that it is the title to the annuity that will commence at the end of the given term, and that the first payment is not to be made till a year afterwards; that is, in the case here specified, not till the end of 16 years.

SCHOLIUM.

The value of the *remainder* of two joint lives, after a given term of years, is likewise the value of £1 due at the end of the given term multiplied by the value of two joint lives, each older by the given term than the given lives; and this product, multiplied by

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[t] If the payments are confined to the beginning of each year, (See note, page 20 5 guineas should be multiplied into 10.18, and the answer will then be £168.55.

M.

the probability, that the given joint lives shall not fail in the given term; or (which is the same) by the product of the two probabilities, that the single lives shall each continue the given term. And the value of an annuity, on any given joint lives for a term of years beginning now, is this last value subtracted from the whole present value of the joint lives. Thus; the value of two joint lives, one 40 years of age, and the other 50, (see Table VII. Vol. II.) is 8.91; which, multiplied by 0.6755, the value of £1 due 10 years hence, and by $\frac{445}{531}$ (the probability that a life at 30 shall continue 10 years) and also by $\frac{346}{445}$, (the probability that a life at 40 shall continue 10 years) give 3.92, the present value of the remainder of two joint lives, aged 30 and 40, after 10 years; and this value, substracted from 10.43, (the value in Table VII. ibid. of two joint lives, aged 30 and 40) leaves 6.51, their value for 10 years.

As the value of the longest of two lives is always the value of the *joint* lives, subtracted from the sum of the values of the two *single* lives; their value also for any *given term*, is the value of the *joint* lives for ~~the~~ given term, subtracted from the sum of the values of the *single* lives for the given term.

The truth of these rules may easily appear without particular proof. I have, however, pointed out the method of demonstrating them in a note[u] at the end of this work.

By similar operations, may be found the values of *three* or more *joint* lives, or the longest of *three* or more lives, for a given term of years, or of what shall remain of them after a given term of years.

QUESTION VII.

"The present value is required of an annuity to be enjoyed by one life, for what "may happen to remain of it beyond another life, after a given term: that is, "provided *both* lives continue, from the "present time, to the end of a given term "of years?"

ANSWER.

Find the value of the annuity for two lives greater, by the given term of years, than the given lives. Discount this value for the given term; and then, multiply by the probability, that the two given lives shall *both* continue the given term; and the product will be the answer.

EXAMPLE.

Let the two lives be each 30. The term seven years. The annuity £10. Interest 4 *per cent.*——The given lives, increased by 7 years, become each 37. The value of two joint lives each 37, is (by Table VII. Vol. II.) 10.25. The value of a single life at 37, is (by Table VI. ibid.) 13.67. The former, subtracted from the latter, is 3.42, or the value of an annuity for the life of a person 37 years of age, after another of the same age, by Quest. I.—3.42 discounted for 7 years, (that is, multiplied by 0.76, the value of £1 due at the end of seven years by Table I. Vol. II.) is 2.6.—The probability that a single life at 30 shall continue 7 years, is (by the hypothesis explained page 2.) $\frac{49}{56}$[x]. The probability, therefore, that two such lives shall both continue 7 years, is $\frac{2401}{3136}$, or, in decimals 0.765. And 2.6, multiplied by 0.765, is 1.989, the number of years purchase which ought to be given for an annuity, to be enjoyed by a life now 30 years of age, after a life of the same age, provided both continue 7 years. The annuity then being £10, its present value is £19.89.

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[u] See note (B) at the end of Vol. II.

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[x] In this case, it is on some accounts best, as well as easiest, to take the probabilities of life from the hypothesis, rather than immediately from the Tables.—Fifty-six persons being supposed alive at 30, one will die every year, according to the hypothesis. At the end of seven years then, the number of the living will be 49, and $\frac{49}{56}$, or the odds of 7 to 1, is, by note, p. 18, the probability, that a life, aged 30, will continue 7 years; and this fraction, multiplied by itself, is the probability, that two lives of this age shall *both* continue 7 years. In general, it must be remembered, that the probability, that any two or more events shall *all* happen, is the product arising from multiplying by one another, the probabilities of all the events taken separately. The probability, therefore, that any number of persons will *all* live any given time, is rightly found by multiplying into one another the probabilities that each of them will live that time. It may further be of use to some, that I should observe here, that the difference between unity and the fraction expressing the probability that an event will happen, gives the probability that it will *not* happen. Thus; the probability, that a person 40 years of age will live 11 years, is, by the *Breslaw* Table, (or Table V. Vol. II.) $\frac{335}{445}$. The probability, therefore, that he will *not* live 11 years, is

By similar operations, it may be found, that supposing the term one year, and the ages and the rate of interest the same, the present value of the same reversionary annuity is £32.4; and that if the term is 15 years, the value is £9.7[y].

For two lives each 40, these values are £30.33.—£17.44.—£7.3, the term being 1, 7, or 15 years.

For two lives each 50, the same values for the same terms, are £28.2—£13.86—£4.34[z].

These values, according to the *London*

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is $\frac{335}{445}$ subtracted from unity or $\frac{110}{445}$. In like manner: The probability that two persons aged 30 shall *both* live 7 years, being 0.765, the probability that they will *not* both live so long, or that *one or other* of them will die in 7 years, is 0.765 subtracted from unity, or .235.

If any reader is unwilling to take these assertions for granted, he should consult the beginning of Mr. *De Moivre's*, or Mr. *Simpson's* Treatises on the Doctrine of Chances, where he will find them demonstrated.

[y] By the *Northampton* Table £10.03, and the six following values by the same Table are £31.01, £17.89, £7.61, £28.66, £14.3, and £4.64. M.

[z] See Note (C) at the end of Vol. II.

Observations and Mr. *Simpson's* Tables of the values of single and joint lives, are,

For 2 lives at 30—*l*.32.05—*l*.18.62—*l*.7.66.
at 40—*l*.30.7 —*l*.15.6 —*l*.5.45.
at 50—*l*.29.36—*l*.12.33—*l*.3.24.

QUESTION VIII.

"Let the scheme of a society for granting
"annuities to widows, be, that if a member
"lives *a year* after admission, his widow
"shall be entitled to a life annuity of £20.
"If *seven* years, to £10 more, or £30 in
"the whole. If *fifteen* years, to another
"additional £10, or £40 in the whole
"What ought to be the annual payments of
"the members for the ages of 30, 40, and
"50, supposing them of the same ages
"with their wives, and allowing compound
"interest at 4 *per cent*?"

ANSWER.

According to the *hypothesis*, explained p. 2; and, therefore, very nearly according to the Tables of Observation for *Breslaw* and *Norwich*, or Tables V. and VIII. at the beginning of Vol. II.

£8.44; £8.69; £9 05.

According to the *London* Observations.

£9.41; £10.17; £10.92[a].

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[a] According to the Northampton Table £8.34; £8.91; £9.44. M.

These values are easily deduced from the values in the last Question. For example. The value of £10 *per annum* for life to 40 after 40, provided the joint lives do not fail in *one* year, is, according to the *hypothesis*, £30.33. The value of £20 *per annum*, in the same circumstances, is therefore £60.66. In like manner, the value of £10 after *seven* years, is £17.44 And of £10 after 15 years £7.3. These values together make £85.4, or the value of the expectation, described in this Question, in a *single present payment*; which, divided by 9.82, the value (by Table VII. Vol. II.) of two joint lives at 40, gives £8.69, the value of the same expectation in *annual payments*, during the joint lives. In the same manner may be found the answer in all cases to any Questions of this kind.

These calculations suppose, that the annual payments do not begin till the end of a year. If they are to begin *immediately*, the true *annual payments* will be, as was before observed, the *single* payments, divided by the value of the joint lives increased by unity; and in the present case they will be, by the *hypothesis*,

£7.75; £7.9; £8.07

By the *London* Observations,

£8.52; £9.00; £9.51[b]

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[b] By the *Northampton* Observations £7.66, £8.03, £8.4. M.

By the method of calculation now explained may be easily found in all cases, supposing the annual payments previously settled, what are the reversionary annuities corresponding to them in value. Thus, the annuities being the same with those mentioned in this Question, the *mean* annual payments for all ages between 30 and 50, are nearly £8 according to the *highest* probabilities of life; £9 according to the *lowest*; and 8 guineas the *medium*[c]; interest being at 4 *per cent.* and the first payment to be made immediately.

If the mean annual payments, beginning immediately, are fixed to five guineas, the corresponding life annuities will be nearly (by the *hypothesis*) £12, if the contributor lives a year, and £24 if he lives seven years; or (by the *London* Observations) £12 if he lives a year, and £20 if he lives seven years[d].

It

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[c] The value of this expectation, supposing married men 40 years of age, and their wives 30, is, in a *single* payment, £113. In annual payments beginning immediately, £9.88, by the *hypothesis*. And £107.—and £10.93, by the *London* Observations. (*b*)

(*b*) By the Northampton Table £116.4 and £10.13.

M.

If the annuities in expectation are £14 provided a member lives a year, and £20 provided he lives seven years, the proper *mean single* payments for all ages, taken one with another, under 50 or 52, is 50 guineas nearly, according to all the Tables of Observation, supposing equality of age between men and their wives. And the addition

It is observable, that the difference in the values of the annuities, arising from difference of ages and the difference in the probabilities of life, is less in this Question than in Question IV; and that, consequently, the plan proposed in it, is the safest, as well as the most equitable and encouraging, that a society can adopt.

It is necessary to remark here further, that *yearly* payments which begin immediately, are more advantageous than *half-yearly* payments which begin immediately. In an Essay published in the *Philosophical Transactions*, Vol. LXVI. p. 109, and inserted in the latter part of this volume, I have shewn that in the case of life-annuities, *half-yearly* payments which begin at the end of half a year, are nearly a *fifth* of a year's purchase better than *yearly* payments which begin at the end of a year. And it is manifest, that

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addition which ought to be made, on account of excess of age on the man's side, is, taking the nearest and the easiest round sums, about a guinea and ½ for every year as far as 17 years; or, in the annual payments, (supposed 5 guineas) ½ a guinea *per annum* for five years excess, and ½ a guinea more for every four years excess beyond five years, till the excess comes to be 17 years. And, I believe, that 60 guineas in *single payments*, and six guineas in *annual payments* beginning immediately, may very well be stated as the *lowest common* payments proper to be required, supposing all married men under 52 taken into a society, without inquiring into the difference of age between them and their wives, the annuities being all along supposed to be *life* annuities, and interest reckoned at 4 *per cent.*

half-yearly payments, which begin immediately, are no more than half a year's purchase better than those which begin at the end of half a year. But *yearly* payments, which begin immediately, are a *whole year's* purchase better than the same payments to begin at the end of a year. The difference of value, therefore, between *yearly* and *half-yearly* payments, supposing both to begin immediately, is three tenths of a year's purchase in favour of the former. The whole of this subject may be seen accurately stated in the essay just referred to.

QUESTION IX.

"The value is required of an annuity to "be enjoyed for what may happen to re-"main of one life after another, provided "the life in expectation continues a given "time?"

ANSWER.

Find by Question VI. the present value of the annuity for the remainder of the life in expectation, after the given time, and multiply this value by the probability, that the other life shall fail within that time. Find also by Question VII. the value of the reversion, provided *both* lives continue the given time. Add these values to one another, and the *sum* will be the answer in a single present payment.

EXAMPLE.

An annuity of £10 for the life of a person now 30, is to commence at the end of 11 years[e], if another person now 40, should be then dead; or, if this should not happen, at the end of any year beyond 11 years in which the former shall happen to survive the latter: What is the present value of such an annuity, reckoning interest at 4 *per cent.* and taking the probabilities of life as they are in Dr. *Halley's* Table, or Table V. at the beginning of Vol. II.?

The value of £10 *per annum*, for the remainder of the life of a person now 30, after 11 years, found by Question VI. is £69.43. The probability that a person 40 years of age shall live 11 years, is, by Dr. *Halley's* Table, $\frac{335}{445}$. The probability, therefore, that he will die in 11 years, is $\frac{335}{445}$ subtracted from unity[f], or $\frac{110}{445}$; which multiplied by £69.43, gives £17.16. The value of the reversion, provided *both* live 11 years, found by Quest. VII. is £17.23. And this value added to the former, makes

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[e] That is, the title to the annuity is to commence at the end of 11 years, and the first payment to be made a year afterwards, in case the life in expectation should continue so long, and the other fail. But if *both* lives should continue the given term, the first payment is always to be made at the end of the year, in which the former life shall happen to survive the latter. See Quest. VI.

[f] See the Note, p. 24.

£34.39

£34.39 the value required in a *single present payment;* which payment divided by £11.43, (the value by Table VII. Vol. II. of two joint lives, aged 30 and 40, with unity added) gives £3[g]; or the value required in annual payments during the joint lives, the first payment to be made immediately.—If, every thing else being the same, the assigned term is 15 years, the value required will be £29 in a *single payment*, and £2.55 in *annual payments*.

QUESTION X.

"What money in hand, or in annual "payments during life, ought a person of "an assigned age to give for a sum of

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[g] See the demonstration of this rule in Note (D) Vol. II. (c)

(c) This demonstration is derived from Mr. *De Moivre's* hypothesis, and therefore I have added in the note above-mentioned another demonstration founded on the real probabilities of life, from which the following rule is obtained, which is rather more concise than the rule given by Dr. Price.—"Find by Quest. VI. the value of "the annuity for the remainder of the life in expectation "after the given time. Find also by the Scholium to "the same Question the value of the annuity for the "remainder of the two joint lives after the given time. "The latter subtracted from the former will be the value "required."—Thus; the value of an annuity of £10 on a life of 30 after 11 years is £69.43—the value of the like annuity during the remainder of two joint lives aged 30 and 40 after 11 years is £35.08. Subtracting this latter sum from the former we have £31.35, as given above, for the value required. By the *Northampton* Table the value of this annuity is £33.77. M.

"money, payable at his death to his heirs[h]?
"In other words, what money in hand, or
"in annual payments during life, ought a
"person of a given age to pay for an *assurance* of any given sum on his life?"

ANSWER.

Subtract the value of the life from the *perpetuity*. Multiply the remainder by the product of the given sum into the interest of £100 for a year: and this last product, divided by £100 increased by its interest for a year, will give the answer in a *single present* payment. And this payment, divided by the value of the life, will give the answer in *annual* payments, during the continuance of the life.

Example. Let the life be 30. The sum £100. The valuation of lives, that in Table VI. Vol. II. and the rate of interest 4 *per cent.* The perpetuity, therefore[i], is 25. The interest of £100 for a year is £4; £100, increased by its interest for a year, is £104. And the value of the life 14.68. The value of the life, subtracted from the perpetuity, gives 10.32, which, multiplied by the product of £100 into 4, or by £400, gives 4128. And this, divided by 104, gives £39.7, the value of £100, payable at the death of a person aged 30, in a single present payment.—And this payment, divided by 14.68, is £2.7, the same value in annual payments during the continuance of the life.

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[h] This Question is the same with Problem 16th, in Mr. *De Moivre's* Treatise on Annuities, and Problem 26th, in Mr. *Simpson's* Select Exercises; but the answers there given are right only when applied to reversionary *estates*, and therefore must be materially wrong when applied to reversionary *sums*, as will appear from the *Scholium* to this Question, and from Note (E) at the end of this Vol.

[i] That is; the value of the *fee-simple* of an estate found by dividing £100 by the rate of interest.

These values found in the same way agreeably to the valuation of lives for *London*, in Table XI. Vol. II. are £45.76, and £3.49. If the life is 36, and interest 4 *per cent.* these values are £43, and £3.1, by Table VI. Vol. II. and £4.1, by Mr. *Simpson's* valuation of lives for *London* in Table XI.—If interest is reckoned at 3 *per cent.* the same values are, by *De Moivre's* valuation of lives for 30 years of age, £48.14, and 2.86—For 36 years of age, 51.43, and £3.28[k].

It appears here, that difference of interest makes no considerable difference in the answers to Questions of this kind, except when the values are required in a single payment.

If the first of the annual payments is to be made immediately, the single payment is to be divided by the value of the life, with unity added to it, agreeably to what has been already observed; and the annual payments in this case (interest supposed at 4 *per cent.*)

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[k] These values by the *Northampton* Table are £47.8 and £2.83, and £51.28 and £3.26. M.

will be by Mr. *De Moivre's* valuation of lives, (or Table VI. Vol. II.) for a life at 30, £2.53—At 36, £2.9.

If the payments are half-yearly payments beginning immediately, the single payment must be divided by the value of the life increased by seven tenths, (see Question VIII. page 29.) And the half-yearly payments, for the age of 36, will be half 2.96, or 1.48. And half 1.48, or .74, is likewise nearly the proper quarterly payments.

Again; if an annual payment, beginning immediately, of £2.9, ought (reckoning interest at 4 *per cent.*) to purchase £100 payable at the failure of a life now 36; £5, by the rule of proportion, ought to purchase £172. And in like manner, it may be found, that the same annual contribution, in half-yearly or quarterly payments, beginning immediately, ought to purchase £170.—These sums, according to the *London* Observations, are £132 and £130 nearly.

The reason of mentioning these particulars will be seen in the next chapter.

SCHOLIUM.

If the reversion is not a *sum*, but an annuity for ever, or an *estate* in *fee-simple*, to be entered upon after a given life, its present value, *in a single payment*, will be "the value "of the life subtracted from the perpetuity, "and the remainder multiplied by the annuity, or the annual rent of the estate."

And the value, in *annual payments*, will be, as before, the single payment divided by the value of the life. Universally. It ought to be remembered, that a reversionary *estate*, after any given life or lives, is worth so much more than a corresponding reversionary *sum*, as £100, increased by its interest for a year, is greater than £100. Thus, the present values, in single and annual payments, of £4 *per annum* for ever, and of £100 in money after any assigned life, are to one another, (interest being at 4 *per cent.*) as 104 to 100, or 1.04 to one. The reason of this difference is, that the calculations suppose, that the reversionary *sum*, and the first yearly rent of the *estate*, or first payment of the annuity, are to be received at the same time, after the extinction of the lives in possession. It is easy to see, that this is a circumstance which must make the latter of most value. But to prevent any doubts about it, I shall explain it more particularly in a note in the Appendix[1].

QUESTION XI.

"A person of a given age, having a "yearly income which will fail with his "life, wants to make provision for another "person of a given age, in case the latter "should happen to survive. What ought "the former to give in a single payment,

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[1] Vid. Appendix, note (E) at the end of this Volume.

" or in annual payments during their joint
" lives for a given sum payable at his
" death to the latter?"

It is manifest, that the value of the given sum in this case, must be less than in the case stated in the last Question; because, here the payment of it is suspended on the contingency, that one life shall survive another; whereas, in the other case, it is *certainly* to be paid at the failure of a given life.

ANSWER.

Find, by the solution of Problem XXXII. p. 297, Mr. *Simpson's* Select Exercises, the value of an estate, corresponding to the given sum, and depending on the given survivorship. Divide this value by £1, increased by its interest for a year, and the quotient will be the value of the given sum in a single present payment. And the single payment, divided by the value of the given joint lives, will be the answer in annual payments during the joint lives.

The solution I have referred to is as follows:

" Find the value of an annuity on two
" equal joint lives, whereof the common age
" is equal to the age of the older of the two
" proposed lives; which value subtract from
" the perpetuity, and take half the remain-
" der. Then say, as the *expectation* of the
" duration

“duration of the younger of the two lives is to that of the elder, so is the said half remainder to a 4th proportional, which will be the number of years purchase to be given for the estate when the life in expectation is the oldest of the two. But if this life is the youngest, then add the number of years purchase just found to the value of the joint lives, and let the sum be subtracted from the perpetuity, and you will also have the answer in this case[m].”

Let

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[m] Mr. *Simpson* has given the following examples of this solution, adapted to *London* lives:——Example I. “Suppose the age of the expectant to be 40, of the *possessor* 30. The rate of interest 4 *per cent.* and the given legacy £5000 or £200 *per annum.* Then the value of two equal joint lives of 40, being 8.1, (see Table XII.) and the perpetuity 25, the remainder or difference will be here 16.9; whereof the half is 8.45. Therefore, it will be as 23.6 to 19.6, so 8.45 to 7.02 years purchase, or £1404, the required value.”

Example II. “Let the age of the *expectant* be 30, of the *possessor* 40, and the rest as in the preceding example. Here the value of the joint lives 30 and 40, will be 8.8; which added to 7.02, (found above) the sum will be 15.82; whence the answer, in this case, is 9.18 years purchase, or £1836.”

I have shewn, that the values of reversionary *estates*, and reversionary *sums*, are not the same as is here supposed. The rule gives the true value when applied to the former; but, when applied to the latter, the values given by it must be divided by £1 increased by its interest for a year, as above directed. The same observation is to be applied to Mr. *Simpson's* next Problem, or the 33d.

In these Examples 23.6 and 19.6, are the expectations, in Table X. of 30 and 40, according to the *London* Ta-

bles

Let the life in expectation be 30; and the other life 40: The sum, £100. Interest, 4 *per cent.* The valuation of lives, Mr. *De Moivre's*, or that in Table VI. Vol. II.

The *expectation* of the first life, is 28; of the second life 23, by Mr. *De Moivre's hypothesis*. The value of the joint lives is 10.43. by Table VII. Vol. II. The value of two joint lives, both 40, is 9.82, by the same Table. The estate corresponding to £100 is £4 *per annum*, and the present value of such an estate to be entered upon by a person 30 years of age, provided he survives a person 40 years of age, is, by the rule just quoted, £33.32. And this value, divided by £1, increased by its interest for a year, or by 1.04, is £32.03, the value in a *single present payment* of the sum of £100, dependent on the given survivorship. And this single payment divided by 10.43, is £3.07, the required value in *annual payments*, during the joint lives, if the first payment is not to be made till the end of a year. But if the first payment is to be made immediately, the required value in *annual*

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bles of Observation; and the method of finding them for any age, and from any Tables of Observation, is explained at the beginning of the first Essay.

In Mr. *De Moivre's hypothesis*, the expectation of a life is always *half* the complement. See note, p. 2. Sometimes the *complement* of a life is mentioned without any view to Mr. *De Moivre's* hypothesis, and it then means double the *expectation* of the life, whatever that may be, according to any Table of Observations.

payments will be £32.03, divided by 11.43, or £2.8. These values, according to the *London* Observations, or Mr. *Simpson's* Tables founded upon them, are £35.30, in a *single payment*; and £3.6, in *annual payments*, beginning immediately.

Mr. *Simpson*, in the Problems following that here quoted, has given solutions of most other Questions, concerning the values of reversions depending on survivorships, where the whole duration of two or three lives is concerned. And I am acquainted with no other solutions of these Questions, which are applicable to all Tables of Observations[n], and which at the same time (proper regard being paid to the correction explained in the last Question)

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[n] The solutions of this and the following Question were certainly the most accurate of all that had been given previous to the fifth edition of this work. About three years before that edition was published, more correct solutions of these Questions were communicated by myself to the Royal Society, and published in the 77th volume of the Philosophical Transactions. These being derived from the real probabilities of life, depend on no hypothesis, and give the exact values in all cases; while the solutions of Mr. *Simpson*, though applicable to any Table of Observations, are deduced from the *expectations* of life, and therefore can be regarded only as *approximations*, even where they happen to be most accurate. In the earlier and latter periods of life, however, they are much too incorrect for use; nor can it indeed be ever necessary to have recourse to them, since the exact values may be obtained with little or no difficulty from the solutions above-mentioned. The general rule for determining the value of the reversion in the present Question may be expressed as follows: "Let F represent a life one year "younger, and P a life one year older than B. Multi-"ply the value of an annuity on the joint lives of A P, "increased by unity, into the probability that B lives a

a year. Multiply the difference between the perpetuity
“ and the value of an annuity on the joint lives of A B
“ into the interest of £1 for a year. Add these two
“ products together, let their sum be divided by £1, in-
“ creased by its interest for a year, and reserve the quo-
“ tient. Divide the value of an annuity on the joint
“ lives of A F, by the probability that F lives a year.—
“ Subtract this from the reserved quotient, and the
“ remainder being multiplied into half the given sum
“ will produce the value required, when B the *expectant*
“ is the oldest of the two lives.” If B be the youngest, the value will be obtained, as in Mr. *Simpson's* rule, by subtracting the value of A's expectation, found above, from the *whole* value of the Reversion after the joint lives of A and B.

EXAMPLE.

Let it be required to determine the value of £100 payable on the death of A, aged 24, should B, aged 67, be then living, computing at 5 per cent. and from the probabilities of life in the Northampton Table of Observations. In this case the ages of F and P will be 66 and 68 years. The value of an annuity on the joint lives of AP, by Table XXVIII. and XXIX. Vol. II. is 5.924, the probability that B lives a year is $\frac{1392}{1472}$, 6.924, (or 5.924 increased by unity,) multiplied into $\frac{1392}{1472}$, produces 6.550. The value of an annuity on the joint lives A B, (or 6.130,) subtracted from 20, the perpetuity, leaves 13.87, which being multiplied into .05, or the interest of £1 for a year, produces .694. Adding this to the former product, and dividing 7.244, (their sum,) by 1.05, we have 6.899 for the quotient to be *reserved*. The value of an annuity on the joint lives of A F, (or 6.3[illegible]) divided by $\frac{1472}{1552}$ (d), (the probability that F lives a year) quotes 6.675, which being subtracted from 6.899, the *reserved*

(d) Or, which is the same thing, *multiplied* into $\frac{1552}{1472}$.

Question) may be considered as sufficiently correct°.

QUESTION XII.

"Suppose an institution for the relief of
"widows to extend its assistance likewise
"to the families of married men, provided
"they leave no widows. Suppose, for in-
"stance, that in this case children are to be
"entitled to £100. What is such an ex-
"pectation worth, in present payment, in-
"terest being at 4 *per cent*?"

ANSWER.

If 40 is the mean age at which members are admitted on such an institution, and 32 the mean age of their wives, the answer (supposing no subsequent marriages) is, by the 33d Problem in Mr. *Simpson's* Select Exercises, p. 298, and the correction already

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reserved quotient, and .224, the remainder, being multiplied into 50, or half the given sum, we have £11.20, for the value required. If A had been 67 and B 24, the foregoing value must have been deducted from 66.05, the whole value of the Reversion after the extinction of the joint lives of A and B, found by Question X. and the remainder, or £54.85 would have been the answer in this case. These values agree nearly with the values by Mr. *Simpson's* rule. But if the difference of age between A and B is very great, or a Table of Observations is used in which the decrements of life are not so regular as in the *Northampton* Table, the values by Mr. *Simpson's* rule will be found in many cases to be *one third* wrong. M.

° See the third Chapter.

explained, £13.80[p], taking the expectations and values agreeably to Mr. *De Moivre's* hypothesis.

But

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[p] This Problem and its solution are given by Mr. *Simpson* in the following words: "A, and his heirs are
" entitled to an estate of a given value, upon the decease
" of B, provided B survives A; to find the value of
" their expectation in *present* money.—Solution. " Find
" the value of an annuity on the longest of two equal
" lives, whereof the common age is that of the older of
" the lives A and B; which value subtract from the
" perpetuity, and take half the remainder; then it will
" be, as the expectation of duration of the younger of
" the lives A and B, is to that of the older, so is the
" said half remainder to the number of years purchase
" required, when the life of B *is the older of the two*. But
" if B *be the younger*; then to the number thus found,
" add the value of an annuity on the longest of the lives
" A and B, and subtract the sum from the perpetuity, for
" the answer in this case."

If the estate is £4 *per annum*, the age of B 40, and of A 30, *interest 4 per cent.*, the answer by this rule comes out £13.86, which divided (as in the preceding Question) by 104, gives £13.32, the value, as above, of £100 in money. If B is 30 and A 40, the same value is £15.71.

N. B. The value of the longest of the two lives is always the *difference* between the value of the *joint* lives, and the *sum* of the values of the two given *single* lives. Thus; the value of a life at 40, is, by Table VI. Vol. II. 13.2. The *sum* of the values of two such lives is 26. The value of two joint lives, whose common age is 40, is, by Table VII. 9.82; and the difference is 16.58, or the value of the *longest* of two lives at 40. (*e*)

(*e*) This rule, like the preceding one of Mr. *Simpson*, being deduced from the *expectation* of life is equally incorrect: and therefore, in most cases, it will be best to adopt the following general rule deduced from the Solution to which I have referred in pag. 40. " Let F, as in
" the former question, denote a life one year *younger*, and

" Pa

But there is a reduction necessary, on account of the chance there is, that a widower may marry again. Suppose, therefore, one half of all widowers to marry a second and third time, and that two fifths of such widowers

“P a life one year older than B —Multiply the value of the joint lives of A P, increased by unity, into the probability that B lives a year. Subtract the difference between the joint lives A B and twice the value of the single life B from the perpetuity, and multiply the remainder into the interest of £1 for a year. Let this be added to the former product, divide their sum by £1 increased by its interest for a year and *reserve* the quotient. Divide the value of the joint lives A F by the probability that F lives a year—subtract this from the *reserved* quotient, and the remainder being multiplied into half the given sum will produce the value required.”

EXAMPLE.

Let it be required to determine the value of £100 payable on the death of B, aged 65, should that happen *after* the death of A aged 25, computing at 3 *per cent.* and from the probabilities of life at *Northampton.* The value of an annuity on the joint lives A P by Tables XXVIII. and XXIX Vol. II. is 7.123 which being increased by unity and multiplied into $\frac{1552}{1632}$ (the probability that B lives a year, found by Table XVII.) produces 7.725. The value of an annuity on the joint lives A B is 7.370.—The value of an annuity on the life of B by Table XIX. is 8.302, and the interest of £1 for a year is .03—The difference between 7.37 and 16.601 (or twice 8.302) being subtracted from 33.333, the perpetuity, and 24 099 the remainder multiplied into .03 produces .723, which being added to 7.725 the former product, and 8.448, their sum being divided by 1.03 we have 8.202 for the quotient to be *reserved.*—The value of an annuity on the joint lives A F is 7.611, the probability that F lives a year is $\frac{1632}{1712}$, the former divided dowers ſurvive theſe ſubſequent marriages. In this caſe, $\frac{1}{2}$ added to $\frac{2}{5}$ of $\frac{1}{2}$, or, $\frac{7}{10}$ of all who become widowers, will die without leaving widows, and therefore $\frac{7}{10}$ of £13.8, or £9.66, will be the anſwer. If only *one fourth* of all who become widowers marry again, and two fifths of theſe ſurvive, the anſwer will be £11.73.

This calculation ſuppoſes all marriages to leave children who ſurvive their parents. If this is conſidered as uncertain, the values now determined muſt be diminiſhed in the proportion of this uncertainty. Thus; if one marriage in ſeven fails of leaving children[q] that ſurvive their parents; theſe values will be reduced a *ſeventh* part, or to £8.28, if *half*, and £10.05, if a *quarter* of all widowers marry.

In this way may any other queſtions of the ſame kind be anſwered on any ſuppoſitions that may be thought moſt reaſonable.

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divided by the latter quotes (*f*) 7.984, which being ſubtracted from 8.202 the *reſerved* quotient, and .218, the remainder, being multiplied into 50, or half the given ſum, we have £10.90, for the value required. Suppoſing A to be 65 and B 25 years of age, the foregoing ſum muſt be ſubtracted from 41.761, the whole value of the reverſion after the longeſt of the two lives of A and B, and £30.861, the remainder, will be the value in this caſe. M.

[q] This for many years has been nearly the fact among the miniſters and profeſſors in *Scotland.*

(*f*) Or, which is the ſame thing, *multiplied* into $\frac{1712}{1632}$.

QUESTION XIII.

"Let an establishment be supposed which "takes in at once all the marriages in a "country, or all marriages among persons "of a particular profession within a given "district, and subjects them for perpetuity "to a certain equal and common tax, or an"nual payment, in order to provide life an"nuities for such widows as shall result from "these marriages. What ought the tax to "be, supposing the annuity £20, and calcu"lating at 4 *per cent.* from Mr. *De Moivre's* "valuation of lives?"

ANSWER.

Since, at the commencement of such an establishment, all the oldest, as well as the youngest marriages, are to be entitled equally to the proposed benefit, a much greater number of annuitants will come immediately upon it, than would come upon any similar establishment, which limited itself in the admission of members to persons not exceeding a given age. This will check that accumulation of money, which should take place at first, in order to produce an income equal to the disbursements at the time when the number of annuitants comes to a *maximum*; and, therefore, will be a particular burden upon the establishment in its infancy. For this, some compensation must be provided;

and the equitable method of providing it, is, by levying *fines* at the beginning of the establishment, on every member *exceeding* a given age, proportioned to the number of years which he has lived beyond that age. But in the present question, it is supposed, that such fines cannot be conveniently levied, or that every payment must be equal and common, whatever disparity there may be in the value of the expectations of different members. The fines, therefore, must be reduced to one common one, answering as nearly as possible to the disadvantage I have mentioned, and payable by every member at the time when the establishment begins. After this, the establishment will be the same with one that takes upon it all at the time they marry; and the tax or annual payment of every member adequate to its support, will be the annual payment during marriage, due from persons who marry at the mean age at which, upon an average, all marriages may be considered as commencing.—There are then two points to be here determined. The *fines* necessary to be paid at first, according to the account I have just given; and the *constant annual payment*, necessary to be made by every member, as an equivalent for the expectation provided by the establishment.—The *fines* to be paid at first are, for every particular member, the same with the difference between the value of the expectation to him at his present age, and what would have been its value to him had the scheme begun at the time he married? Or, they are, for the whole body of members, the difference between the value of the common expectation, to persons at the mean age of all married persons taken together as they exist in the world, and to persons at that age, which is to be deemed their mean age when they marry.

Thus; let 33 for the man, and 25 for the woman, be the mean ages of all that marry annually. Let also 48 be the mean age of all the married men in the world, and 40 of married women*.—Now, he that will calculate for these ages, in the manner directed in Quest. IV. will find, that the value in *annual payments* during marriage, and beginning immediately, of the expectation of an annuity of £20 *per annum* by a person 25 years of age, after a life whose age is 33, is £6.64.—And that £8.04, is the value of the same expectation, the ages being 48 and 40.

The former, therefore, is the payment for perpetuity from every member of the establishment; and the value of the *difference* between it and the latter, or of £1.4. *per ann.* payable during two joint lives, whose ages are 40 and 48, that is, £14.2, is the fine necessary to be levied on every married member at the beginning of the establishment[s].

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* I must beg leave to refer to note (F) in the Appendix, for an explanation of what I mean by the mean ages of married men and women, and also for a confirmation of the answer I have given to this Question.

It would be easy to extend the benefit of such an establishment, so far as to provide £100 for the children of members, provided they leave no widows; and the necessary addition on this account to the perpetual annual payments can scarcely, in the circumstances this question supposes, be much more than about 15*s.* payable during life, and excluding from all benefit such as happen to be widowers at the commencement of the establishment, and do not afterwards marry.

If, in such an establishment, all persons of a particular denomination, whether married men, widowers, or bachelors, are subjected alike to the taxes and fines; they ought to be as much *less*, as the whole number of persons subjected to them is *greater* than the number of marriages constantly existing.

In carrying these schemes into execution, there cannot be a more easy, or equitable way of raising the necessary fines, than by providing, that none shall be entitled to any

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[s] An annuity for ever, the first payment of which is to be made immediately, is worth 26 years purchase, interest being at 4 *per cent.* £14.2 therefore is equivalent in value to £0.55 or 11*s.* *per annum*, for ever. Add this to £6.64, and it will appear, that £7.19 *per annum*, beginning immediately, is the answer to this Question, supposing the value of the *fine* to be provided for in the perpetual annual payments.

expectation for a few of the first years. Thus, an establishment, entitling widows to £20 *per annum* for life, and consisting of 667 married members, and 344 unmarried, always kept up at an average, ought to begin with a capital of £14.2 multiplied by 667, or £9471, besides one payment in hand of the constant annual payments. That is, (the proper annual payment of every member being in this case, $\frac{667}{1011}$ multiplied by £6.64, or £4.38) it ought to begin with a capital of £13,899 over and above the payment of £4.38, at the *end* of every year for ever afterwards[1].—The exclusion of all the first members from any benefit, unless they survive the first *two* years, or live to make *three* payments, would raise this capital nearly. And such an exclusion for *three* or *four* years, would be an advantage so considerable, that it would probably give security and stability to the scheme for all subsequent time.

In these observations, I have had in view some schemes which have been established in this kingdom; but more particularly, one established by act of parliament among the clergy in *Scotland*; of which I shall have occasion in the next chapter to take further notice.

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[1] Or, supposing the value of £9471 (the fine) provided for in the annual payments, it ought to receive every year, at the *beginning* of the year, a contribution from each member of £4.74.

I have chosen to calculate here only from Dr. *Halley's* Table, or Mr. *De Moivre's hypothesis* grounded upon it, because the *London* Table is, by no means, adapted to the cases in view.

It should be further remembered, that when the mean ages, at which marriages commence, are supposed to be 33 and 25, all second and third marriages are included; and that it is to be expected, that almost all these marriages will begin after these ages; and likewise, that a considerable proportion of the first marriages will begin a much longer time *after* these mean ages, than any of the other first marriages will begin *before* them.—Probably, therefore, these mean ages should not be taken younger. One or two years, however, more or less, in every supposition I have made, will make no difference of any consequence.

QUESTION XIV.

" A person of a given age has an estate de-
" pending on the continuance of his life for
" a given term. What ought he to give for
" having it *assured* to him for that term?"

ANSWER.

From the value of an annuity certain for the given term, found by Table II. subtract the value of the life for the given term, found by Quest. VI. and *reserve* the remainder.—Multiply the value of £1 due at the end of the given term, (found by Table I.) by the *perpetuity*, and also by the *probability*, that the given life shall fail in the given term. The *product* added to the *reserved* remainder, and the *sum* multiplied by the given annuity, will be the required value of the assurance in one present payment[a].

EXAMPLE.

An estate or annuity of £10 *for ever*, will be lost to the heirs of a person now 34, should his life fail in 11 years. What ought he to give for the *assurance* of it for this term?—That is; What is the present value of such an annuity to be entered upon at the failure of such a life, should that happen in 11 years.

The value of the life of a person whose age is 34 for 11 years, is, by Quest. VI. (reckoning interest at 4 *per cent.* and taking Mr. *De Moivre's* valuation of lives) 7.76; which, subtracted from 8.760, (the value of an annuity certain for 11 years) leaves 1 the remainder to be reserved.

The value of £1 to be received at the end of 11 years, is, 0.6496, by Table I. Vol. II. The probability that the life of a person, aged 34, shall fail in 11 years is, by Dr. *Halley's* Table, $\frac{102}{199}$; and the perpetuity is 25.

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[a] See the demonstration in note (G) at the end of this volume.

These numbers multiplied by one another, and 1 added to the product, make 4.32, which, multiplied by 10, (the given annuity) gives £43.2, the required value in a single present payment.

£43.2 divided by 1.04, gives £41.54 the true value (by Scholium to Question X.) of the assurance of an *equivalent sum*, or of £250 for 11 years on the given life[x].

Again

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[x] The premium of assurance for a given *sum* may be more easily obtained by the following rule: "Find the "value of the life for one year less than the given term. "Let this value added to unity be divided by £1 in"creased by its interest for a Year. From the quotient "subtract the value of the life for the given term, and "the remainder being multiplied into the given sum will "be the value required."

The equivalent annual payments, the first to be made immediately and the others at the beginning of each year, may be found, "by dividing the single premium by the "value of the life, with unity added, for one year less "than the given term." In the rules given by Dr. *Price*, in this question, and the scholium, the first payment is supposed to be made at the beginning, and the last at the expiration of the *term*. But this method of assurance is never adopted; the annual payments being always made, according to the rule given above, at the beginning of each year.

EXAMPLE.

Let it be required to determine the value of an assurance of £250, in single and annual payments, for a term of 11 years on the life of a person aged 34, according to the *Northampton* Table of Observations, and at 4 *per cent.* interest.—The value of the life for 10 years is 7.335, and 8.335 (or 7.335 increased by unity) being divided by 1.04 quotes 8.014; from which deducting 7.851, (or the value of the life for 11 years) we have .163; which

Again 41.54, divided by 8.76, (the value of the given life for the given time with unity added to it) gives 4.74, the same value in annual payments beginning immediately, for 11 years[y], subject to failure should the life fail.

SCHOLIUM.

In a similar way may the price of assurances on any two joint lives, or the *longest* of two lives for any given terms, be calculated; the rule being as follows:

"From the value of an annuity certain
" for the *given term,* subtract the value of
" the joint lives, or the longest of the two
" lives for the *given term,* found by Scho-
" lium to Quest. VI. and *reserve* the remain-
" der.—Multiply the value of £1 to be re-
" ceived at the end of the given term by the
" perpetuity, and also by the probability
" that the *joint lives,* or the *longest of the two*
" *lives,* shall fail within the given term.
" This product added to the reserved *re-*
" *mainder,* and the *sum* multiplied by the
" annuity to be assured, will be the value of
" the assurance in a single present payment."

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which being multiplied into 250, produces £40.75 for the value of the assurance in a single payment.—And 40.75 divided by 8.335 gives £4.89 for the same value in annual payments to be made at the beginning of each year.

M.

[y] The last payment to be made at the end of the 11th year; or 12 payments in all.

EXAMPLE.

EXAMPLE.

"What is the value of £10 *per annum*, "to be entered upon, should *either* of two "persons, one 40, and the other 30 years of "age, die in ten years, reckoning interest "at 4 *per cent.* and calculating from Dr. "*Halley's* Table."

The value of two joint lives at these ages, for 10 years, (found by *Scholium* to Quest. VI.) is 6.51; which, subtracted from 8.11, (the value of an annuity certain for 10 years, at 4 *per cent.*) leaves 1.60, the remainder to be *reserved*.

The value of £1 to be received at the end of 10 years, is .6755, by Table I. Vol. II.

The probability, that the lives of one or other of two persons, aged 30 and 40, shall fail in 10 years, is $\frac{185}{531}$ by Table V.[z] And the perpetuity 25. These numbers, multiplied by one another, and 1.60 added to the product, make 7.48, which multiplied by 10, (the given annuity) gives £74.8, the answer in a single present payment.

£74.8, divided by 1.04, gives £71.92, the value of the assurance of an *equivalent*

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[z] The probability taken from the Table, that a person, aged 30, shall live 10 years, is $\frac{445}{531}$. That a person, aged 40, shall live 10 years, is $\frac{346}{445}$. That they shall *both* live 10 years, is $\frac{346}{445}$, multiplied by $\frac{445}{531}$, or $\frac{346}{531}$. That they shall *not both* live 10 years, or that *one* or *other* of them shall die in this time, is $\frac{346}{531}$, subtracted from unity, or $\frac{185}{531}$. See note p. 24.

sum; or of £250; £71.02, divided by 7.51, (the value of the two joint lives for 10 years with unity added) gives 9.57, the value of the same sum in annual payments beginning immediately, for 10 years, subject to failure should the joint lives fail.

EXAMPLE II.

"What is the value of £10 *per ann.* to "be entered upon, should two persons, one "30, and the other 40, *both* die; that is, "should the *longest* of the two lives fail in "10 years, reckoning interest at 4 *per cent.* "and calculating from Dr. *Halley's* Table?"

The value of the *longest* of the two lives for 10 years, (that is, the value of the joint lives for 10 years, subtracted from the sum of the[a] values of the single lives for 10 years) is 7.91; which, subtracted from 8.11, the value of an annuity certain for 10 years, leaves .20 the remainder to be reserved.—The value of £1 to be received at the end of 10 years, is, .6755. The probability that the lives of two persons, aged 30 and 40, shall fail in 10 years, is, by Table V. $\frac{86}{531}$ multiplied by $\frac{99}{445}$ or $\frac{8514}{236295}$; and the perpetuity 25. These numbers, multiplied by one another, and .20 added to the product, make .740, which, multiplied by 10, (the given annuity) gives 7.4, the answer in a single payment.

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[a] See Scholium to Question VI.

7.4, divided by 1.04, gives 7.11, the value of the assurance of £250.

REMARK I.

The values of single lives for given terms, when these terms are less than ten years, must, in answering these Questions, and also in answering the following Questions, be found true to at least 2 or 3 places of decimals. When they cannot be found to this exactness by any Tables, they must be calculated in the following manner:

“ Multiply the probability, taken out of “ the Table of Observations, that the life “ shall exist 1, 2, 3, &c. years, by the value “ of £1 due at the end of 1, 2, 3, &c. years; “ and the sum of the products will be the “ value of the life for 1, 2, 3, &c. years.”

For Example. The probability, that a person whose age is 34, shall live a year, is, by Dr. *Halley's* Table, $\frac{490}{499}$. The probability at the same age, of living 2 years, is, $\frac{481}{499}$; 3 years, $\frac{472}{499}$—$\frac{490}{499}$ multiplied by .9615, (the value, by Table I. of £1 due at the end of a year, interest being at 4 *per cent.*) is, .942; or the value of the life for *one* year—$\frac{481}{499}$, multiplied by .925, (the value of £1 due at the end of two years) is .891. And this added to the former product, gives 1.833; or the value of the life for two years.—$\frac{472}{499}$ multiplied by .8890, (the value of £1 due at the end of 3 years,) is .841; and this product, added to 1.833, makes 2.674, or the value of the given life for 3 years.

When the term exceeds 10 years, the rule in Quest. VI. will give these values with sufficient exactness; and it would do the same in all cases, were the values of lives given true to 3 or 4 places of decimals[b], and in strict agreement to the Tables of Observation used.

The remark now made is to be extended to the values of *joint* lives for given terms. For these values, like those of *single* lives cannot be found in solving these Questions with sufficient accuracy, (when the terms are small, and the values of lives are given only to one or two places of decimals) by any method, except the tedious one, of multiplying the probability that the 2 lives shall *both* continue, 1, 2, 3, &c. years, by the value of £1 due at the end of 1, 2, 3, &c. years, and taking the sum of the products in the manner just described.

Such tables are given in the present edition of this treatise; and therefore this remark is now less necessary than it was. See the tables of the values of single and joint lives in the next volume, deduced from the register of mortality at *Northampton*, &c.

REMARK II.

If the annuity is to be entered upon, in case of the failure within a given time of any life or lives, *at the end of that time;* and not *at the end of the year in which the failure may happen;* its present value will be the product arising from the continual multiplication by one another of the perpetuity increased by unity, the value of £1 due at the end of the given time; the annuity, and the probability that the life, or lives, shall fail within the given time. And care should be taken not to confound these two sorts of Questions with one another.—Thus, the value in one payment of £10 *per ann.* to be entered upon eleven years hence, in case a person aged 34 should not live so long, is 26, (the perpetuity increased by unity, interest being at 4 *per cent.*) multiplied by .6496, and by £10 and also by $\frac{103}{499}$; or 34.8.—This value, divided by 1.04, is 33.5, the value of an equivalent sum, or of £250 to be obtained on the same conditions.

The value of the *assurance* of any annuity on the whole continuance of any single life, is, by Quest. X. the *excess* of the perpetuity above the value of the life, multiplied by the annuity. And in like manner, the value of the *assurance* of any annuity on the whole continuance of any two *joint* lives, or the *longest* of two lives, is the excess of the *per-*

petuity above the value of the joint lives, or of the longest of two lives, multiplied by the annuity. This is very obvious; but no general method has been yet explained of finding the values of *assurances* on lives and survivorships for terms of years less than the whole continuance of the lives. For this reason, I have been here more explicit than I should otherwise have been; and as such assurances are now much practised, and may be very useful if their values are rightly determined, I have thought proper to add the two following Questions, which, when joined to Question XI. and Mr. *Simpson's* 32d *Problem* given in the note p. 38, will, I believe, exhaust this subject as far as two lives can be concerned.

QUESTION XV.

"B, expectant, will lose a given sum, "should he survive A, *within a given time*. "What ought he to pay for the *assurance* of "it?"—In other words: "What ought he "to pay for a given sum to be received at "the death of A, should he happen to sur"vive him within a given time?"

ANSWER.

Divide the *sum* of the decrements of life in the Table of Observations from the age of A, for the given time, by the given time; and, by the *quotient*, divide the number of the living in the Table at the age of A; and again, by this second quotient[c], divide the given sum, reserving the *third* quotient.

Find the value of an annuity on the life of B, for the given time. To this value add the *quotient*, that will arise from dividing the value of an annuity certain, for the given time, by twice the *complement* of the life of B; and the *sum*, multiplied by the *reserved quotient*, will be the required value in a single present [d]payment[e].

EXAMPLE.

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[c] When the age of A is under 60, and the term so large as to exceed the difference between it and 70, it will be best, when the *London* Table is used, to divide the given sum, not by the second quotient here mentioned, but by the *complement*, or double the expectation of A.

[d] See the demonstration of this rule, and also of the rule that will be given for solving the next Question, in note (H) at the end of this volume.

[e] The following rule derived from the solution of the 11th Problem (see note G) is rather more simple, and is so far preferable to the rule given above, as it has been obtained without the aid of Mr *De Moivres's* hypothesis, which it is best entirely to exclude from the doctrine of Survivorships. "Divide the sum of the decre-"ments of life from the age of A for the given time, "by the given time multiplied into the number of the "living at the age of A, and *reserve* the quotient. Let "the value of the life of B for one year less than the "given term, with unity added, be divided by £1, in-"creased by its interest for a year. To the quotient add "the value of an annuity on the life of B for the given "term, multiply this quantity into the *reserved* quotient "and also into half the given sum, and the product will "be the value required."

EXAMPLE.

EXAMPLE.

Let the Table of Observations be Mr. *Simpson's* for *London* (see the Tables in the next volume). Let the rate of interest be 3 *per cent.* A, seven years of age. B, 30. The given time 14 years. The given sum £100. —The sum of the *decrements,* for 14 years

EXAMPLE.

Let the respective ages of A and B be 7 and 30 years, the given sum £100, the term 14 years, the rate of interest 3 *per cent.* and the probabilities of life as they are in the *Northampton* Table.—The sum of the decrements for 14 years from the age of 7 is 865; the number of persons living at that age is 5925, which being multiplied into 14, and 865 divided by 82950 (the product) we have .0104 for the quotient to be *reserved.* The value of an annuity on the life of B for 13 years is 9.436, and 10.436 (or 9.436 added to unity) divided by 1.03 (or £1 increased by its interest for a year) quotes 10.132; which being added to 9.937, the value of an annuity on the life of B for 14 years, and 20.069 (the sum) being multiplied into .0104, (the *reserved* quotient) and also into 50, (half the given sum) we have 10.4 for the value required.—The same value by the *London* Table is £11.94, or £1. 2*s.* more than Dr. *Price* makes it to be. But it should be observed, that when the decrements of A's life for the given term are equal (as they happen to be very nearly by the *London* Table in the present case) that the exact value of the reversion is obtained by this rule, and therefore that the value by Dr. *Price's* rule in this instance is *one tenth* less than the truth. This inaccuracy arises from applying Mr. *De Moivre's* hypothesis to a Table of Observations for which it is by no means suited. Had the *Northampton,* or any other table been used, which agrees better with this hypothesis, the values by both rules would have been nearly the same. M.

from the age of seven, is 75, which, divided by 14, gives 5.36. The number of the living at seven is 430, which, divided by 5.36, and £100 divided by the quotient, gives 1.12, the *quotient* to be *reserved.*

The value of an annuity for 14 years on the life of B, is, by Quest. VI. 9.54.—The value of an annuity certain for 14 years, is, (by Table II. Vol. II.) 11.296, which divided by 94.4, (twice the *complement* of the life of B, by Table X.[f] gives .12, which, added to 9.54, gives 9.66; and this again multiplied by 1.12, the *reserved quotient*, gives 10.82, the *present* value in *one* payment of £100, payable at the death of A aged 7, to B aged 30, should A die and leave B the survivor within 14 years.

The present value for 14 years of two joint lives, one 7 and the other 30 years of age, may be found, by the help of Table XII. and the rule in the *Scholium* to Quest. VI. to be nearly 9 years purchase; and, £10.82 divided by this value with unity added, or by 10, gives 1.082, the foregoing value in *annual payments* during the joint lives for 14 years, the first payment to be made immediately, and the *last* payment at the end of 14 years, should the joint lives not fail.

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[f] This table gives the *expectations* only, but it should be remembered, that twice the *expectation* is always the *complement* of a life. See note, p. 39.

SCHOLIUM.

It deserves particularly to be remembered, that in this method likewise may be calculated, what sums ought to be paid on any survivorship, within a given time, of one life beyond another, in consideration of any given sum now advanced.—The following Example of this is a case which has offered itself in practice.

"A person, aged 30, has in expectation "an estate which is to come to him, pro-"vided he survives a *minor*, aged 7, before "he is out of his minority; that is, pro-"vided he should be himself living at the "time of the minor's death, should that hap-"pen before he is 21.—In these circum-"stances, he wants to borrow £1000 on his "*expectation*. What *reversion* out of the "estate depending on such a survivorship, is "a proper equivalent for this sum now ad-"vanced, interest being reckoned at 3 *per* "*cent.* and the probabilities of life being "supposed the same with those in Mr. *Simp-*"*son's* Table of *London* Observations?"

ANSWER.

It appears from what has been just determined, that for £10.82 now advanced, the proper equivalent in such circumstances, is, £100 to be paid, in case the survivorship should take place; or, by the *correction* in page 35, as much of the estate as £100 will buy at 3 *per cent*. supposing the first rent to be received immediately; (that is, supposing the estate worth 34.35 years purchase) or £2.912 *per annum*.—By the rule of proportion, therefore, for £1000 the proper equivalent will be £9242 in money, or £269 *per annum* out of the estate.

QUESTION XVI.

"£100 will be lost to B's heirs, should "he happen to die after A, *within a given* "*time*. What is the price of the *assurance* "of it?—That is: What is the present va-"lue of £100 payable at the death of B, "provided his death should happen *after* A's "death, *within a given time?*"

ANSWER.

Divide the sum of the decrements of life in the Table of Observations from the age of B, for the given time, by the given time; and by the *quotient* divide the number of the living at the age of B; and again, by this *second quotient*[g], divide the given sum, reserving the *third quotient*.

Find the value of an annuity on the life A for a number of years, less by *one* year than the given time, which subtract from the value of an annuity certain for the same number of years. Multiply the *remainder* by the *reserved quotient*, and divide the *product* by the amount of £1 for one year, and let this be a *second* reserved quotient.

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[g] Or rather, if the *London* Table is used, by the *complement* of the life of B, when his age is under 60, and the term exceeds the *difference* between it and 70.

Again. Multiply into one another the *first* reserved quotient, and the value of an annuity certain for the given time; and divide the product by twice the *complement* of A's life. This *last* quotient, added to the *second* reserved quotient, will be the *answer* in a present single payment[h].

EXAMPLE.

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[h] The solution of the 12th question (see note G at the end of this volume) affords an easier and in general a more correct method of determining the value of this reversion: "Divide the sum of the decrements of life from the age "of B for the given time by the given time multiplied into "the number of persons living at the age of B, and *re-* "*serve* the quotient. Divide the difference between the "value of an annuity certain, and the value of an an- "nuity on the life of A for one year less than the term, "by £1 increased by its interest for a year. Add the "quotient to the difference between the value of an an- "nuity certain and the value of an annuity on the life of "A for the given term. Let this quantity be multiplied "into the *reserved* quotient, and also into half the given "sum, and the product will be the value required."

EXAMPLE.

Let the respective ages of B and A be 40 and 30. The sum £100. The rate of interest 4 *per cent.* The term 20 years; and the probabilities of life as they are in the *Northampton* Table.—The sum of the decrements of life in this table from the age of 40, for 20 years, is 1597, which, divided by 3635 multiplied into 20 (or by the number of persons living at the age of B multiplied into the given term) gives .022 for the quotient to be *reserved*.—The value

EXAMPLE.

Let the age of B be 40. Of A 30. The sum £100. Rate of interest 4 *per cent.* The given time 20 years. The Table of Observations, Mr. *Simpson's*, or Table IX. in the next volume.—The sum of the decrements of life, in this Table from the age of 40 for 20 years, is 127, which, divided by 20, (the given time) gives 6.35.—The number of the living at 40 is 229, which, divided by 6.35, gives 36.06; and £100 (the given sum) divided by 36.06, gives 2.77, the *first* quotient to be reserved.

The value of an annuity for 19 years on a life at 30 years of age, is 10.45; which, subtracted from 13.134, (the value of an annuity *certain* for 19 years, by Table II.)

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value of an annuity certain for 19 years is 13.134. The value of an annuity on the life of A for the same time is 11.134. The difference, or 2, divided by 1.04 (or £1 increased by its interest for a year) quotes 1.923 The value of an annuity certain for 20 years is 13.590. The value of an annuity on the life of A for the same term is 11.432. The difference, or 2.158 being added to 1.923, makes 4.081; which. multiplied into .022, the *reserved* quotient, and also into 50, half the given sum, produces £4.189 for the value sought.—By the *London* Table this value is £7.71, which agrees nearly with the value by Dr. *Price's* rule; and this will always be the case when the lives are between 20 and 50 years of age. But if one of the lives be very young and the other very old, this latter rule, when the *London* Table is used, will be found exceedingly incorrect. If the computation, however, be made from the *Northampton*, or any other table agreeing better with Mr. *De Moivre's* hypothesis, the values, as in Question XV. will be nearly the same by both rules. M.

and the remainder multiplied by 2.77, gives 7.435. This product divided by 1.04, (the amount of £1 in one year) gives 7.15; the *second* reserved quotient.

2.77 multiplied by 13.59, (the value of an annuity certain for 20 years) gives 37.644; and this *product* divided by 94.4, (twice the complement of A's life, by Table X.) gives .399, which, added to 7.15, gives £7.55 the *Answer;* or, the value of £100 payable at the death of B, on the contingency of his surviving A aged 30, and *both* dying in 20 years.

It is plain, that this is likewise the sum that ought to be lent to B now, on the expectation of £100, at his death, provided it should happen after A's death in 20 years.

This rule gives the just solution in all cases, except when B, the expectant, is the *youngest* of the two lives, and at the same time the term of years *greater* than the complement of A's life. In this particular case the following rule must be used.

Find, by the preceding rule, the value of the assurance of the given sum for a term of years, equal to the complement of A's life, and let this value be *reserved.* Multiply by one another the given *sum;* the *value* of £1 to be received at the end of a number of years equal to the complement of A's life; and the value of an *annuity certain* for as many years as the given term exceeds this complement: And the *product*, divided by the complement of B's life, and the *quotient* added to the *value reserved*, will be the true value sought.

EXAMPLE.

Let the age of B be 30; of A 40. The term 47 years; and every thing else as in the last Example. The complement of A's life, is, by Table X. 39.2. The value of £100 to be received at the death of B, if he survives A within 39 years, may be found by the preceding rule to be £16.15; the value to be reserved—The value of £1 to be received at the end of 39 years is, by Table I. .2166. The value of an annuity *certain* for 8 years, (the excess of the given term above the complement of the life of A by Table X.) is, 6.733.

And these two values multiplied by one another, and by £100 give 145.83; which, divided by 47.2, (the complement of the life of B) and 16.15, added to the quotient, make £19.23, the value sought.

REMARK.

As after finding the present value of an estate, or annuity, it is necessary to *divide* that value by the amount of £1 in one year; in order to find the present value of a *sum* equivalent to the *annuity*; so, after finding the value of a sum, it is necessary to *multiply* that value by the said amount, in order to find from it the value of an equivalent annuity.

In the first example, therefore, the value of an estate of £4 *per annum* would be £7.85. In the second example £20. And this is, as it ought to be, the value for the whole duration of the lives, agreeably to the Problem in the note p. 38[i].

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[i] The value of a term, if it fall short of the difference between the age of B and that of the oldest life in the Table of Observations, cannot agree exactly with the value for the whole duration of life. Indeed it seldom happens that the value of an assurance (unless it be extended to the continuance of both lives) is required for such a term as shall exceed the limit of A's life. If this case, however, should occur, its solution may be obtained from the following rule: "Find by the preceding rule (see note p. 66) the "value of the assurance for a term of years equal to the "difference between the age of A and that of the oldest "life in the Table, and let this value be *reserved*. Find "by Question XIV. the value of the assurance of the "given sum on the life of B for the whole term, and also "for a term which shall be equal to the difference between "the age of A and that of the oldest life in the Table. "Subtract the latter from the former; and the remainder "being added to the value *reserved* will be the answer."

EXAMPLE.

Let the ages of B and A respectively be 30 and 56. The term 48 years. The sum, the rate of interest, and the Table of Observations, the same as in the Example in note page 66.—The difference between the age of A and that of the last life in the Table is 40. The value of £100 to be received on the death of B *after* A in 40 years is £17.13, which is the sum to be *reserved*.—The value of an assurance on the life of B for 48 years, is 37.69. The value of the same assurance for 40 years is 35.06. The difference, or 2.63, being added to 17.13, the sum *reserved* above, In solving this Question, care also must be taken not to forget the *first* Remark under the foregoing Question.

In this chapter, rules have been given for finding the values of all assurances on single lives, and any two lives, or any survivorships between two lives, whether for *terms*, or their *whole duration*. In the same way rules may be investigated for finding the values of all assurances on any three lives, or any survivorships between them. But this is a work of more difficulty, and which requires great attention and skill. I can, however, with particular satisfaction acquaint the Reader, that it has been lately executed, in the compleatest manner, by Mr. Morgan, in his Treatise on the *Doctrine of Annuities and Assurances on Lives and Survivorships*[k].

CHAP.

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above, we have £19.76 for the value required.—By Dr. *Price's* rule this value is £19.55. Had the assurance been extended to the whole duration of the lives, the correct value would have been 21.59.—Hence it appears that although the term is rather longer than in the Example given by Dr. *Price*, yet that the value of the assurance is by no means the same with its value for the whole life. The difference, however, is in this case rather too great by both rules, but in neither of them is the error considerable. M.

[k] These solutions to which Dr. *Price* refers, having been derived, like all which preceded them, from the *expectation*, and not from the *real probabilities* of life, are liable to the objections which I have already noticed, (see note, p. 40). I have, however, since the publication of the above Treatise, given correct solutions of all the Problems involving three lives in the survivorship; which the Reader may see in

CHAPTER II.

Containing an Application of the Questions in the foregoing Chapter to the Schemes of the Societies in Great Britain, *for making Assurances on Lives and Survivorships, and for granting Annuities to Widows, and to Persons in Old Age.*

SECT. I.

Of the London Annuity, *and the* Laudable *Societies for the Benefit of Widows*[1].

THE scheme mentioned in Quest. VIII. was nearly that with which the *London Annuity* Society set out in 1765. The *Laudable Society* was established in 1761, and formed on a similar plan. In both, the *annual contribution* of every member was five guineas, payable half yearly; and for this a title was given to an annuity of £20 to every widow during widowhood, if the husband after admission, lived *one* year according to the *first* scheme; or *three* years according to the

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in the several volumes of the Philosophical Transactions for the years 1789, 1791, 1794, and 1800; and the general rules, deduced from these solutions, are inserted at the end of this volume. M.

[1] It must be remembered, that this section has in view the state of these societies in 1771, or at the time of the publication of the first edition of this tract.

the[m] *second*; of £30, if the husband lived *seven* years, according to both schemes; and £40 according to the *first* scheme, if he lived 15 years, or 13 years, according to the second. —In both schemes also, there was no other premium or fine required, than five guineas extraordinary, at admission, from every member whose age does not exceed 45. The *Laudable Society* admitted none above 45, and the *London Annnuity* Society obliged every person between 45 and 55 to pay, at admission, five guineas extraordinary, for every year that he was turned of 45.

These were the main particulars in the schemes on which these Societies were formed; and, therefore, both of them, were the annuities to be enjoyed for life, received (supposing the members all under 46 at admission, and of the same ages with their wives, and money at 4 *per cent.*) but little more than three-fifths of the true value of the annuities; or about one half, supposing wives, one with another, 10 years younger than their husbands; as appears from Question VIII.

It appears further in that Question, that, supposing the annuities to be *life* annuities, and men and their wives, of equal ages, the expectation to which an annual payment of five guineas beginning immediately, entitles, is nearly £14 if the contributor lives a year, and £20 if he lives seven years [n], taking the medium between the *London* and the other Tables of Observations.

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[m] In this society a member who lived but *one* year, was entitled to no more than an annuity of £10 for his widow; if he lived two years, to £15; if he lived three years, to £20; four years, £25; seven years, £30; ten years, £35; thirteen years, £40.

It is likely, that many persons will be very unwilling to believe, that these schemes could have been so deficient as they have been now represented. I will, therefore, endeavour to prove this in a way which, though less strict, is sufficiently decisive, and may be more likely to be intelligible to persons unskilled in mathematical calculation.—I shall here confine myself to the scheme of the *London Annuity* Society. The differences between it and the scheme of the *Laudable* Society are inconsiderable, and what shall be said of the one will be fully applicable to the other.

According to this scheme, as it has been just described, all that live 15 years in the society will be entitled to annuities of £40 *per annum*, for their widows. Suppose the whole society, at admission, to be men of 40 years of age, taken one with another. A person of this age has an even chance of *living* 23 years; and he has an even chance of continuing with a wife of the same age, (that is, of continuing in the society) $13\frac{1}{2}$ years°. Not much less, therefore, than half the members will continue in the society 15 years; and, consequently, not much less than half the widows that will come upon the society will be annuitants of £40 *per annum*. These widows, however, being older than the rest when they commence annuitants, will continue on the society a shorter time; and, therefore, the number constantly in life together, to which they will in a course of years increase, will be proportionably smaller. Putting every thing as favourably as possible, let us suppose, that out of 20 annuitants constantly on the society, *five* will be annuitants of £40, *six* of £30, and *nine* of £20. To 20 annuitants then the society will pay £560 *per annum*, or the 20th part of this sum, that is £28 to *every* annuitant at an average. But such an annuity for a life at 40, after another equal life, provided both survive one year, is worth, (by Quest. VII. p. 23.) in a single present payment, £85 nearly, according to the *London*, and all the

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[n] 'The same annual payment will, on the same suppositions, entitle to £14 if a member lives a year, and £18 if he lives *three* years.

° This is the exact truth according to Mr. *De Moivre's* hypothesis, and the *Norwich* Table. But according to Dr. *Halley's* and the *Northampton* Table, a man 40 years of age has an even chance of living no more than 22 years, and of joint continuance with a wife of the same age. 13 years.—Forty must be more than the mean age of the members of the society at admission, and on this account the number of annuitants of £40 must be proportionably greater. The mean age, therefore, has been taken very moderately.

Tables of Observations, interest being all along supposed at 4 *per cent.*

It cannot appear improbable to any one, that this should be the true value of such a reversion. It is not probable, that there is any situation in which the decrements of life are such as can make it a tenth part more or less. £85 in present payment is the same with £3...8*s. per annum* for ever.—But is an annual payment of five guineas, which must cease as soon as either of two lives each 40 fails, equal in value to such a perpetuity? Every one must see, that there is a great difference.—A set of marriages between persons all 40, will, according to the probabilities of life in Dr. *Halley's* Table, last, one with another, 15 years[p]; and an annual payment beginning immediately, during the joint continuance of the lives of two persons of this age, is worth 10 years purchase[q].

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[p] See the beginning of Essay I.

[q] The value of such an annual payment, by Table XII. or the *London* Observations, is 9.1; and 10.8, by Mr. *De Moivre's* hypothesis.—I have not taken into this account the five guineas *fine* paid at admission, because it is obviously of too little consequence to make any considerable difference. The allowances I have made in favour of these schemes are more than equivalent to it. In particular, it should be remembered, that the calculations suppose, that the payments required by these schemes, are yearly payments beginning immediately; (see p. 29.) and that the first payment of the annuity is not to be made till the end of the year in which the husband shall die; and also, that the annuity is to be paid yearly, and nothing to be due for any part of the year in which the annuitant shall happen to die.

The comparison then, in the present case, is between £3...8*s*. *per annum for ever*, and five guineas *per annum for* 15 *years*; or between an annuity of £3...8*s*. worth 25 years purchase, and an annuity of five guineas worth only 10 years purchase.

But to throw this subject into another light; Let the number to which the society is kept up be supposed to be 200. It has been demonstrated in Quest. II. that at least half this number of widows will in time come to be constantly on the society; and it has also been just now shewn, that the medium of annuities payable to them, will be at least £28. After a course of years, then, the society will have a constant expence to bear of £2800 *per annum*.—But what will be its income?—In order to determine this, we must consider, that there are two sources from whence its income will be derived. First, the annual payments of the members. And, secondly, the money accumulated, or the *capital* raised during the time the number of annuitants is coming to a *maximum*.—The first of these sources affords 1000 guineas, or £1050 *per annum*. This wants £1750 of the annual expence just mentioned; and, therefore, in order that the income of the society may be equal to the burden upon it, when the annuitants come to a *maximum*, there must be a fund raised in the mean time equal to £43,750, or to an estate in perpetuity of £1750 *per annum*.—But £1050 *per annum* beginning immediately and forborn 25 years, and improved, without loss or delay, all that time at 4 *per cent.* compound interest, will but just raise such a capital[r]. There is therefore, the fullest proof that the scheme I am considering is extremely deficient. The truth is, that scarcely a *third* of such a capital could be raised, as will appear from the following observations.

Out of 200 persons, all 40 years of age, *more* than five, according to the *London* Table of Observations, and not so many by Dr. *Halley's* Table, may be expected to die in a year. Suppose then five to be the real number of members that will die in the first year of the society. In subsequent years the collective body of members will be continually growing older; and, therefore, the proportion of them that will die every year, will be continually increasing, till it gets to a *maximum*. I will, however, suppose, that during the first 20 years no more than the number just specified will die every year; and that, consequently, no more than *five widows* will come every year on the society. The ages of all these widows, when they commence widowhood, will, it is evident, be between 40 and 60. One with another then, they may be considered as having commenced widowhood at 50 years of age. Now, five widows left every year at this age, will, in 10 years, increase to 43 constantly in life together, according to the expectations of life in Tables V, VIII, and XVII; and, in 20 years, to 70[5]. Suppose the true number alive together at the end of 20 years to be only 62. The greater part of these will be annuitants of £30 and £40 *per ann.* and the rest £20. Were the former only equal to the latter, the medium of annuities payable to them would exceed £25. Suppose then this medium to be no more than £26, and it will follow, that, at the end of 20 years, the society will have an annual rent to pay of £26 multiplied by 62, or £1612; and, if then able to bear such an expence, it must, in the intermediate time, have acquired an increase of income equal to the difference between £1050 and £1612 *per ann.* That is; it must, with its savings, have accumulated a stock equal to £562 *per ann.* and worth £14,050. But as, during this time, there will be a number of annuitants constantly increasing, to whom yearly payments must be made, the savings cannot certainly be one half of what they would have been had the society been all the time free from all burdens. Suppose then the stock produced by these savings, to be equal to the stock that would arise from an income of £1050 *per ann.* beginning immediately, and improved perfectly at 4 *per cent.* compound interest, for half the time I have mentioned, or for 10 years, without being subject to any checks or deductions: Such an income thus improved, would in 10 years produce an additional income of £504 *per annum*, or a capital of £12,600.—According to these Observations, therefore, the annual income of the society at the end of 20 years, and before a third part of the highest annuitants could come upon it, would fall considerably short of its expences. About that time then it would necessarily run aground; and long before the number of annuitants could rise to 100, it would spend its whole stock, and find itself under a necessity of either doubling the annual payments of its members, or of reducing the annuities one half.

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[r] Every question of this kind may be easily solved by Table IV. in the next volume, which shews, that £1000 *per annum* will, in 25 years, increase to £2666 *per annum*; and, therefore, £1050 *per annum* to £2800 *per annum*.

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[5] Every calculation of this kind is easily made by the rule in note (A) at the end of this volume—I have put the number living together at the end of 20 years at 62, not only that the reader may be better satisfied that I have kept low enough, but also to make an allowance for such widows as will be left by those members who die within a year after admission, and who, therefore, according to these schemes, will be entitled to no annuities. This allowance is too large: For, after the first year of the scheme, it will not happen above once in 4 or 5 years, that the death of a member will be so circumstanced, supposing the probability that a man at 40 will live a year, to be, as all but the *London* Tables make it, 50 to 1.

All I have now said is meant on the supposition, that the society begins with 200 members at 40 years of age, and is afterwards limited to that number, by admitting no more new members than will just supply the vacancies occasioned by the loss of old members If it is allowed to increase, it may continue a longer time. And, for this reason, a society that wants half the income necessary to render it permanent, may very well subsist, and even prosper for 30 or 40 years.—Thus, the *Laudable Society*, was it to keep to its present number of members, might possibly feel no deficiencies for 20 years to come; but if it should continue to increase at the rate of 70 or 80 every year, it would, at the end of that time, possess a balance so much in its favour as might enable it to support itself for 15 or 20 years more[t]. But bankruptcy would come at last, and

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[t] What has been before demonstrated in Question III. should be here recollected, that the number of annuitants on such a society as this, must go on to increase for more than 100 years after acquiring its greatest number of members.

The *Laudable* Society, I am informed, took its rise from a calculation contained in a pamphlet entitled, *The Possibility and Probability of a* SCHEME *intended for the Benefit of Widows being able to support itself.* The scheme here referred to, is the same with that which this Society has and with the more weight the longer it had been deferred.

The rule in the *London Annuity* Society, which obliges every person between the ages of 45 and 55, to pay at admission 5 guineas extraordinary for every year that he exceeds 45, is an advantage to it, but it is a very inadequate, and also a very unequitable advantage. For at the same time, that it obliges a person 55 years of age, to give *more* than the value of his expectation, it takes *above* two-fifths *less* than the value from a person who is 45 years of age.

If any persons remain still doubtful about what I have said, I must beg their attention to one further argument.

It must be expected that every other member of these societies, supposing them to consist of persons all of the same ages with their wives, will leave widows to whom, one with another (as already shewn), at least £28

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has since followed; and I am afraid I shall not be credited, when I say, that the calculation to prove its capacity of supporting itself, is founded on the supposition, that a hundred married men, whose common age is 36, will leave but one widow every year, though at the same time it is supposed that two of them will die every year.

This mistake has made the whole calculation one half wrong.—Nothing can be plainer than that, if the death of a married man does not leave a widow at the end of the year, the reason must be, that both himself and his wife have happened to die in the year. But it is always very improbable this should happen.

per ann. must be allowed, for as many years as there have been payments from each member. For every 10 guineas then received they must some time or other hereafter pay £28. But let it be well considered what can enable them to do this. Did money bear no interest, for any given sum now received, they could not afford at any time hereafter to pay more than an *equal* sum. That is; (since the duration of *survivorship* is in the present case, by Question II. equal to the duration of *marriage*) the proper consideration for any given reversionary annuity, to be allowed to *all* the survivors of a set of marriages, would be, an equal annuity payable by each marriage during its existence; and just *half* the reversionary annuity, if it is to be allowed only to half the survivors, or to widows exclusive of widowers. The annual payment then of *five* guineas, during marriage, can entitle widows to no more than an annuity of *ten* guineas, supposing money to bear no interest. But if money does bear interest, the same payment will entitle them to more, in proportion to the degree in which it is capable of being improved, during the time between that in which the annual payments begin, and the commencement of widowhood. Now, it is easy to see, that unless money bears very high interest, this improvement cannot be likely in any circumstances to produce a capital, the interest of which shall be equal to the annual

payment.

payment itself. Any given annual payment perfectly improved at 4 *per cent.* compound interest, requires 17 years and a half to double itself, supposing the first payment made immediately[u]. But no marriages are *likely* to last so long, except those among persons who are very young. A marriage between two persons, both 40, will not *probably* last longer than 15 years, according to the probabilities of life in Dr. *Halley's* Table. A marriage between two persons, both 50, will not probably by the same Table, last longer than *eleven* years; nor a marriage between two persons, both 30, longer than 16 years. Such marriages, it is true, may possibly last 30 or 40 years. But this circumstance is more than balanced by the fact, that no less possibly they may not last *one* year. The annual payments, then, being incapable of such an improvement as shall produce an additional income equal to themselves; it is obvious, that no society ought to go so far as to allow to widows annuities twice as great as those which might be allowed, supposing no interest of money[x]; so far, for instance, as to allow, instead of 10 guineas, 20 guineas for an annual payment of five guineas. In the circumstances of most of these societies three-fifths addition may be the full allowance. That is; supposing the annual payment of each member to be five guineas, time may be expected for gaining from hence a capital of 75 guineas, or that shall produce three guineas *per annum* interest; and the proper reversionary annuity will be 16 guineas; or six guineas more than the proper reversionary annuity, did money admit of no improvement[y].

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[u] At 3 *per cent.* the period of doubling money by compound interest, is nearly 23 years and a half. At 5 *per cent.* 14 years.

[x] The money accumulated will not be exactly the same [illegible] that to which the annual payment would increase, if improved at compound interest for a number of years, equal to that which the joint lives have an *equal chance* of existing. Much less will the increase be the same

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same with that which would arise from the annual payment forborn, and improved, for a number of years equal to the *expectations* of the joint lives. It will be less than either of these, for a reason explained in note (K) at the end of this volume.

[y] To these accounts may be added the following short and easy method of trying the sufficiency of all schemes of this kind.

In an adequate scheme it can make no difference whether the annuities themselves are paid, or the value of them in a single payment at the time they become due.—Suppose then a society just established, consisting of 600 members, all married men at the age of 40, each of whom, besides one payment in hand, is to make an annual payment of five guineas. Suppose the age of their wives 39½, and every widow to be entitled, on the day her husband dies, to a life annuity of £20, the first payment to be made at the end of half a year.—Suppose further, that the society is to be kept up for ever to 600 members by admitting new ones at the age of 40, as old ones drop off.—In the first year (according to Tables V, VIII, and XVII, in the next volume) twelve members, at least, will

The preceding observations have gone on the supposition, that the reversionary annuities are to be *for life*. What difference in favour of these societies arises from the circumstance, that the annuities are to be paid only *for widowhood*, cannot be exactly determined. Some judgment, however, may be formed of it from what has been said at the conclusion of Quest. II. Were even one half of the widows to marry, still the schemes I have been considering would probably be insufficient. But, in the circumstances of these societies,

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will die, and leave twelve widows, each intitled to £20 *per annum*. The value of such an annuity to commence at the end of half a year, the age being 40, is 14½ years purchase by Mr. *De Moivre's* valuation of lives, (or Table VI Vol. II.) reckoning interest at 3 *per cent*. The value, therefore, of 12 such annuities; that is, the whole amount of the sums becoming payable during the course of the first year, is £3480.—The annual contribution is 600 times 5 guineas, or £3150, and this, together with its interest for about half a year, or £3205, is all that such a society could be possessed of to bear an annual expence of £3480. It appears, therefore, that, in order to support the expence of the supposed annuities, the annual contribution of each member ought to have been more than five guineas.

A proof of the same nature with that here given may be deduced, by considering these societies as bodies of men united for the purpose of assuring to one another, from year to year, annuities for their widows; and the way of finding the value of such an assurance is, to multiply the value of the annuity, by the probability that it will become payable in the course of the year. For instance, Let the member's age, and also his wife's, be 40. Let the annuity be £20 *per ann.* for life to commence at the end of a year, or an annuity whose present value is (reckoning societies, it cannot be expected, that above one in 10, or perhaps one in 20, will marry. The persons most likely to enter into them, are such as have not the prospect or ability of making competent provisions for their widows in other ways. The widows left, therefore, will in general be unprovided for, and, being also left with families of children, it is quite unreasonable to expect, that any considerable proportion should marry. This is true of such as may happen to be left young; but when a society has subsisted some time, the *greater* part will not be young when left, and these, at the same time that no advantage can be expected from their marrying

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(reckoning interest at $3\frac{1}{2}$ *per cent.*) 14 years purchase; that is, £280. The probability that a person at the age of 40 will die in a year, and that his wife of the same age will live a year: or, in other words, the probability, that such a member will leave a widow in the course of the year, is, by the *Breslaw* Observations, (or Table V, next volume) $\frac{9}{445}$ multiplied by $\frac{436}{445}$, or .0198. (See page 18, and 21.) That is; there will be the odds of nearly 49 to 1, against such a member leaving a widow in the course of the year. The value of the assurance, therefore, is .0198, multiplied by 280, or the 50th part of £280: that is, £5...11*s*. In the same manner the value of a like assurance for a year at any other ages may be easily calculated. At the age of 35, it is £5...7*s*. At the age of 45, it is £6...7*s*. The value, therefore, increases continually with age; and, if given in an annual payment constantly the same, which is the case in these societies, it ought to be greater than the annual payment due for one year at the commencement of the assurance.

Five guineas *per annum*, therefore, is, *demonstrably*, an insufficient payment from a married man for a life-annuity of £20 to his widow.

will be in general the *highest* annuitants, and, therefore, the *heaviest burdens.*—Moreover, the prospect of the loss of their annuities will have a particular tendency to check marriage among them—For all these reasons it seems to me likely, that the benefit, which these societies will derive from marriages among their annuitants, will not be very considerable; or, at least not *so* considerable as to be equal to the advantages I have allowed them, by calculating on the suppositions, that the money they receive will be *always improved perfectly, without loss or delay, at the rate of* 4 *per cent. compound interest;* that the probabilities of life among males and females are the same, and all husbands likewise of the same ages with their wives, and that consequently the *maximum* of widows on such societies can amount to no more than half the number of marriages[1].—With respect to the last of these suppositions, it deserves to be particularly observed, that from accounts taken annually with great care in *Scotland,* it appears, that the widows of the *ministers* and

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[1] Care should be taken in these societies, not to judge of the proportion of widows that will marry, from the proportion that may happen to marry during their first years. For most of the widows that will be left at first will be young; whereas the greater part will not be young when they commence widowhood, after a society has cubsisted 30 or 40 yerrs; and, therefore, though one in three or four should marry at first, it will not be reasonable to expect, that half so many should marry after the affairs of the society become stationary.

professors there[a] notwithstanding the diminution occasioned by their marrying, do exceed considerably half the number of marriages. And certainly it would be unreasonable in these societies not to reckon that the same will happen among them.—Indeed it seems certain, that notwithstanding the hazards that attend child-bearing, the probability that the woman shall survive in marriage, and not the man, is much greater[b] than is commonly imagined. It will be shewn in the second Essay, that it is not less than the odds of 3 to 2; and had I calculated agreeably to this fact, the values of annuities for widows, would have been given near a quarter greater than they have been given on the supposition, that the chance of survivorship is equal between men and their wives.—It must be added, that I have made no account of any expences attending the execution and management of the schemes of these societies. Some such expences there must be, and some advantages should be always provided in order to compensate them.

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[a] The number of married ministers and professors, for 17 years, from 1750 to 1766, was at a medium 667. And from the enquiries that have been made, it appears, that from this whole body near 400 widows constantly living are derived. The medium of widows left annually has, for the last 36 years, been $19\frac{1}{10}$; and, for 10 years, ending in the year 1767, but nine of these had married.—Of the annuitants likewise (about 160 in number) on the fund established among the Dissenters in *London*, for relieving the widows of indigent ministers, it is found that few ever marry. See the 2d Section of this Chapter. See likewise the latter end of the 2d Essay; and note (A) at the end of this volume.—In the *Laudable* Society during 19 years from 1761, the number of widows that came upon the Society was 167; and of these only 13 had married at Lady-day 1780. Fourteen had died.

[b] Partly, as observed in page 8, on account of the greater mortality of males, but chiefly on account of the excess of age on the man's side.—The *Laudable Society*, for several years after its institution, paid no regard to this excess of age; and the allowance required on this account by the *London Annuity* Society was so trifling as to deserve no notice.

In March 1780, 167 husbands had died in the *Laudable Society*, and only 138 wives.

There are in this kingdom many institutions for the benefit of widows, besides the two on which I have now remarked; and in general, as far as I have had any information concerning them, they are founded on plans equally inadequate, having been formed just as fancy has dictated, without any knowledge of the principles on which the values of reversionary annuities ought to be calculated. The motives which influence the contrivers of these institutions, may be *laudable*; but they ought, I think, to have informed themselves better. This appears sufficiently from what has been said; but I will just mention one further proof of it.

The *London Annuity Society* promises that, if in 21 years; and the *Laudable Society* that, if in 25 years, it shall appear that there has been all along an annual surplus in favour of the societies, it shall be employed in either raising the *annuities*, or in sinking the *annual* *payments*. Now, they may be assured, that if at the end of these periods, they should not be possessed of a considerable surplus, the true reason will be, their having granted much higher annuities than the annual contributions are able permanently to support. For it has been demonstrated, that the number of annuitants, and consequently the amount of the annual expences, will go on increasing for a long course of years beyond these periods. The effect, therefore, of carrying into execution this regulation will be, precipitating that bankruptcy which would have come too soon had there been no such regulation.

It has been said in defence of these societies, that the deficiencies in their plans cannot be of much consequence, because their rules oblige them to preserve a constant equality between their income and expences, by reducing the annuities as there shall be occasion. And from hence it is inferred, that they can never be in any danger of a bankruptcy.— But it has appeared, that the time when they will begin to feel deficiencies is so distant, that it will be too late to remedy past errors, without sinking the annuities so much, as to render them inconsiderable and trifling. All that is given too much to *present* annuitants is so much taken away from *future* annuitants. And if a scheme is very deficient, the first annuitants may, for 30 or 40 years, receive so much more than they ought to receive, as to leave little or nothing for any who come after them. Deficient schemes, therefore, are attended with particular injustice; and this injustice will be the same, if, instead of *reducing* the annuities, the annual payments should be increased; for all the difference this can make will be, to cause the injustice to fall on *future contributors*, instead of *future annuitants*.

But what requires most to be considered here is, that, after either the annuities have been for some time in a state of reduction, or the contributions in a state of increase, it will be seen that these societies have gone upon wrong plans, and, therefore, they will be deserted and avoided; the consequence of which will prove still greater deficiencies in their annual income, and a more rapid desertion and decline, till a total dissolution and bankruptcy take place.—This will be the death of most of the present societies for providing for widows, if they continue to be encouraged, and do not soon alter their plans: And at that period the number of *annuitants* will be greater than ever; whose annuities, having no other support than the poor remains of a stock always insufficient, will be soon left, without the possibility of relief, to lament that ignorance and credulity which gave rise to these societies, and which had so long supported them.

In the *London* Annuity Society, there is an encouragement to *bachelors* and *widowers* to join them, arising from the additional annuities to which they will be *immediately* entitled, when they marry, in consequence of having made their payments a greater number of years: and it is imagined, that particular advantages will be derived from such members. But even these will in general pay much less than the value of their expectations.—A person who begins an annual contribution of five guineas at the age of 24, will, should he live 11 years, and marry a woman of the same age at the end of that time, entitle her immediately to £35 *per ann.* during survivorship, and to £41 *per annum* should he live four years after marrying, (interest being at 4 *per cent.*)[c] In this particular case, therefore, a person will pay nearly the true value of his expectation. But *all* at all ages who *marry*, in less time than 11 years after admission, will pay less than the value of their expectations.

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[c] The value of five guineas *per annum* (first payment made immediately) for 11 years, subject to failure should a life now 24 fail; and, after 11 years, for the joint lives of two persons both 35, is, by the Table of *London Observations*, £69.3—By Dr. *Halley's* Table, £76.44—The present value of £35 *per annum* for life to the widow of a person now 24, should he live 11 years, and marry a woman of the same age with himself at the end of that time; and also of £6 more, or £41 *per annum* in all, should he live after marriage four years; is, by the Table of *London Observations*, £69.36.—By Dr. *Halley's* Table, £76.03.

SUPPLEMENT *to the preceding Section; containing a further Account of the Societies for the Benefit of Widows.*

Of the LONDON ANNUITY SOCIETY.

IN the preceding Section, the reader has seen on what very incompetent plans the two Societies, which are the subject of it, have been formed. Some changes have taken place in them since the last edition of this Treatise, of which it is necessary I should here take notice.

The LONDON ANNUITY Society, consisting in January 1781 of 326 members, has so far reformed its plan, as to be now in little danger. Besides ordering a compensation for difference of age between husbands and wives, it determined, in 1774, not to engage to pay a higher annuity than £20 to widows if their husbands had lived a year after admission, for a contribution of 10 guineas in hand, and five guineas *per ann.* afterwards. At the same time, however, room was left for expecting that some additional annuities might be paid to the widows of such members as should survive 15 years in the Society; but what the additions should be, was left to be determined at the end of 15 years from the establishment of the Society. Accordingly, last year some able judges were consulted; and, if I am rightly informed, the result has been, that the Society has agreed to promise for the same contributions an addition to the £20 annuity just mentioned, of £4 *per ann.* to widows, if their husbands have been members 15 years or more.

He that will consider the demonstrations in the foregoing Section, or compute agreeably to any table of the decrements of life by the Rule in Question X. may assure himself that a contribution of ten guineas in *immediate payment* and five guineas every year after the first, is scarcely a sufficient support for an annuity of £20 during life to widows, supposing husbands and wives of the same ages, and money improved at an interest of 4 *per cent.*—But money may now be improved at a higher interest. Some advantages also must be derived from making the annuities payable for widowhood only; and on these accounts, such a contribution may safely enough be reckoned a proper payment for an annuity of £20 as it is offered by this Society. But it cannot, without danger, offer more; particularly, as it is certain, that the lives of women in general, and more especially of women in the advanced stages of life, are more durable than the lives of males.

The additional annuity, however, not being of more value than about three guineas and a half in a single payment at entrance, the Society may possibly find itself capable of paying it, *provided* the contributions for supporting the scheme (namely ten guineas at entrance and five guineas *per ann.* from every member, besides a just compensation for the excess of his age above that of his wife) are not loaded with any of the expences of management[d]

SECT.

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[d] In the year 1790 this Society referred the examination of their affairs to Dr. *Price* and myself, at which time their number consisted of 328 members, and 85 widows claiming annuities to the amount of £1967—their stock invested in the 3 *per cents.* amounted to £70,500; the dividends on which together with the annual premiums, exceeded their expenditure by £2000 a year.—In consequence of our joint opinion and advice, it was then determined to increase the annuities to widows £1 for every year that the husband should live beyond 24 years, so that if they lived to make 50 payments the annuities should amount to £50.—Since that period another investigation took place in 1798, when it appeared that the Society consisted of 302 members and 132 annuitants claiming £3294 *per annum*—that their stock in the three *per cents.* amounted to £93,000, that their annual income exceeded their expenditure by more than £1000, and, on the supposition that this capital was worth 75 *per cent.* or the money nearly which had been paid for it, that they possessed a surplus of £11,735.

Whatever objection may be made to this method of estimating their capital, it is evident, if the money had been invested where it could suffer no diminution, that the affairs of the Society would have appeared to be in a very flourishing condition. The public funds, however, since the year 1798, have so improved in value, that if the stock were now sold it would produce nearly the sum which was originally paid for it, and therefore no reasonable doubt can be entertained at present of the Society's

Further Account of the Laudable *Society for the Benefit of Widows.*

THIS Society affords a melancholy proof of the pernicious tendency of that disposition to form annuity societies which prevailed some time ago.—In consequence of a petition to Parliament in 1774, from many of the most respectable members, it *reformed* its plan; but no arguments could engage the majority of the members to consent to a reformation which was likely to be attended with any other effect than an increase of calamity by postponing it. For thirteen years from the time of its establishment, it had overlooked the differences of age between men and their wives, and gone upon the plan mentioned in page 72.—In 1774, a

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ciety's stability. Indeed it is not possible that they should ever fall into any serious error, provided that at certain and not very distant intervals, they continue those investigations which have been twice repeated within the last twelve years.

It may not be improper to add, that on the 1st of January, 1802, the members of this Society amounted to 300, the annuitants to 134—the capital to £102,000 stock in the 3 *per cents.* the annual income, including dividends and premiums, to £4804, and the annual expenditure, including annuities and management, to £3374, or £1430 less than the income. M.

compensation for the wife's inferiority of age was ordered to be paid by all *new* members, and at the same time the following plan agreed to.—For an annual payment of five guineas, the first to be made immediately, every widow was entitled to an annuity during widowhood of

£10 if her husband had been admitted			2 years and a day
£15 if	—	—	3 years
£20 if	—	—	6
£25 if	—	—	8
£30 if	—	—	11
£35 if	—	—	13
£40 if	—	—	15

Any one who will calculate by the rule in Quest. VIII. will find that the annual payment necessary to support these annuities is nearly, by Dr. *Halley's* Table of Observations, £7, supposing equality of age between husbands and wives, money improved at 4 *per cent.* and the mean age of admission *thirty-seven*——This change, therefore, did not deserve the name of a *reformation*; and an attention to the following account will shew, that instead of doing good, it has in fact only prolonged the existence of the Society to do mischief.

From the establishment of the Society in 1761 to 1772, it had increased to 700 members; but in April 1780, it had gradually sunk, by deaths and desertions, to 550.—The whole number of widows which had come on the Society was then 168, of whom 84 had come upon it in six years, from 1774 to 1779[e]; that is, fourteen annually. *Thirteen* had died and *fourteen* had married, which had left 141 annuitants, the claims of 133 of whom amounted then to £3,310 *per ann.* The claims of the remaining eight, reckoned at £30 *per ann.* each, will make £240 *per ann.*; and the expences of management (or £300 *per ann.* nearly), added to these sums, will make the whole annual expence of the Society in April, 1780, £3,850. ——Its income, consisting of the interest of £49,090 three *per cent.* stock, and the subscriptions of 550 members at five guineas each, amounted at the same time to £4,360, leaving a favourable balance of only £480. Supposing the Society to preserve its present number of members, and the number of annuitants to increase for six years to come only at the rate of ten annually, its expence at the end of the present year, or the beginning of the next, will be equal to its income; and afterwards it will find itself under the necessity of having recourse to one of the three following expedients. It will be obliged either to run into its capital, or to increase its contributions, or reduce the annuities.—The consequence of the first of these expedients will be, that the capital of the Society will be soon consumed, and the annuitants left without support.—The consequence of the *second* will be, that the contributions will be increasing every year till, in 10 or 12 years, they are *doubled*, and at last almost *tripled*.—The consequence of the *last* will be, that the annuities will sink every year till they come to be less than half the annuities promised[f]. Such are the affairs of this infatuated Society; nor is it easy to apply any remedy to them; for in consequence of going on too long with an insufficient scheme, the Society has large payments to make in order to compensate past deficiencies; and a scheme at first adequate would now prove inadequate.—For example, (supposing a just allowance required for the wife's inferiority of age) an annual payment of seven guineas from every member, begun in 1772, when the number of widows was only 42, would probably have been sufficient to support the reformed scheme mentioned in page 98; but *now* an annual payment of nine guineas from the *present* members, and of seven from all *future* members, would scarcely be sufficient.

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[e] In 1780 fourteen more widows came upon the Society.

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[f] The annuitants (should the number of members continue what it is) cannot increase to much less than double their number last year.

In such circumstances it seems best to break up, and to divide the present capital, as far as it will go among the annuitants. Should this be done, the annuitants will indeed be great sufferers; for, so miserably circumstanced is the Society, that its whole stock will not pay much more than *half*[g] the value of the annuities. But this only sets in a stronger light the necessity of an *immediate* dissolution: I say *immediate*; for the annuitants are increasing fast, together with the medium of the annuities due to them; and therefore the consequence of delay must be, extending *greater* sufferings to *greater* numbers.

POSTCRIPT.

SINCE the preceding account was written, this Society, convinced at last of its mistakes;

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[g] The mean age of the widows now on the Society is not probably more than 42 or 43. The value of annuities payable half-yearly during the lives of women at this age, is not really so little as 13 years purchase, reckoning interest at 4 *per cent*. Supposing therefore the Society to break up some time or other before the end of this year (1781), and the number of widows then upon it 154, the value of their annuities, at £350 each, will be £53,900, of which their stock, at its present price, will not pay so much as £28,000.

mistakes, has resolved to reduce the annuities of such widows as became claimants before the alteration in 1774, 35 *per cent.* and the annuities of all the other widows who are now claimants, 20 *per cent.* It was also resolved by two general courts, that the annual payments should be increased from five to six guineas. But, this resolution was revoked by the general court in July last; and a resolution substituted in its room to change the plan described in p. 98, into one which makes it necessary that a member should have been admitted 3 years to entitle his widow to an annuity of £10, and 7 years to entitle her to £20; and 13 and 20 years to entitle her to £30 and £40.—Possibly this resolution, like the former, will be retracted by some future general court. Should it be confirmed, it will, in conjunction with the reduction just mentioned of the annuities payable to present claimants, constitute a reformation which (supposing the funds of the Society not incumbered with any expences of management) might have been *nearly* sufficient *ten or twelve years* ago to save the Society. But it is far from being sufficient *now*——The reduction of the annuities payable to present claimants is too little. It should have been extended to the widows of the old members now living, and the new plan restricted to members *lately* admitted, and to *future* members; and a compensation for past over-payments to widows should have been provided[h].

SECT.

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[h] This unfortunate Society, since the period in which the above postscript was written, has repeated its attempts to reform those errors in which it originated. In the year 1790 the number of its members had sunk to 330, while the number of its annuitants had increased to 217. Its annual income also had fallen short of the expenditure by £1740, and its capital, after an accurate examination of its finances, appeared to be insufficient to satisfy the demands upon it by £30,000 nearly. Under these circumstances it was determined that the annuities payable to the widows who were then claimants should be reduced *one half*, and that the annuities to all future claimants should be reduced *one fifth*. At that time the Society possessed £42,000 stock in the *four per cents*; and had these continued at par, or at the same price as they were then estimated, it is probable that the Society would have been spared the mortification of any further reductions. But in consequence of the disastrous war which soon ensued, this stock, together with all other property in the public funds, lost more than *one third* of its value—the capital of the Society was diminished in the same proportion, and the annuities, which were yearly increasing upon it, required so much larger quantity of stock to be sold in order to discharge them than the computations supposed in 1790, that new alarms were produced, and it became necessary once more to have recourse to retrenchment. In the year 1802 therefore another investigation of the Society's finances was undertaken, on the supposition that its stock was then worth only 80 *per cent.* or *one fifth* less than it was really worth at the period of the former investigation, and the result was such as might have been expected. The value of the present and future claims was found to exceed the value of the capital by £2257, so that no alternative remained but either to break up the Society, or to reduce the annuities. The former was deemed impracticable, and consequently it was determined that the annuities of the present widows, as well as those which should hereafter become claims upon the Society, should be reduced

SECT. II.

Of the Association among the London *clergy, and the Ministers in* Scotland, *for providing Annuities for their Widows*[1].

IN April, 1765, the clergy within the bills of mortality and the county of *Middlesex*, at a general meeting in *Sion-College*, agreed to form themselves into a society for the support of their widows and orphans. Many in this respectable body may be capable of doing, in a better manner, what I have attempted in this Treatise; and they are, perhaps, already sensible of the deficiencies in the plan which they have established. I shall not however, I hope, do wrong, in taking the liberty to recite briefly this plan, in order to introduce a few observations upon it.

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duced *one sixth*; so that instead of £24, only £20—instead of £16, only £13...6*s*...8*d*. *per annum*, &c. are to be paid in future. The depreciation of the public funds is an event for which no computation can provide, and therefore, as the Society's affairs in their present state depend as much on the stability of those funds as on the lives of its members, it is impossible to say how far this last reduction may prove effectual. The number of its widows amounted in 1802 to 216, but the number of its members scarcely amounted to half their number in 1790. If circumstances therefore should have still continued unfavourable, the next measure must be, the dissolution of the Society, and a division of the remaining capital among the annuitants and surviving members in proportion to their respective interests in the funds of the Society. M.

[1] This section, as well as the former, must be considered as written in 1771. Subsequent accounts of the different institutions are continued respectively at the end of these two Sections.

According to the printed articles, every clergyman possessed of any benefice, lectureship, or licensed curacy, within the bills of mortality and the county of *Middlesex*, who subscribes annually one guinea, or two guineas, or more, shall entitle his widow to an annuity; or, if he leaves no widow, he shall entitle any such children as he shall leave, to the same annuity for seven years as his widow would have had. And, in case a widow possessed of an annuity, should either *die* or *marry* before the lapse of 10 years, from the commencement of her annuity, such children of her former husband as shall be then alive, are to be entitled to as many of the ten years payments of the annuities as she shall not have received.—The annuity is fixed to no particular sum, but, instead of this, it is ordered, that a fourth part of the annual subscriptions and interest shall be divided the first three years after the establishment of the society; half only the next four years; and three fourths the next five years; provided, however, that in no one of these 12 years the dividend shall exceed £20 to the widows and orphans of the clergy subscribing two guineas or more; and £10 to the widows and orphans of the subscribers of one guinea. And, after the expiration of 12 years, the whole amount of the subscriptions, and of the interest of the capital stock, is to be divided proportionably for ever.—It is further provided, that every clergyman who shall be married, or have children, at the time of his subscription, shall pay a fine of two guineas towards a capital stock, if a subscriber of two guineas or more, and 40 years of age or upwards. If 50 years of age or upwards, he shall pay a fine of three guineas; if 60 or upwards, five guineas. But, if not married at the time of his subscribing, and shall afterwards marry, he shall pay a fine according to the age he shall be of at the time of his marrying. The obligation laid upon all, whether married or unmarried, to become subscribers, is, an incapacity of being admitted members without the consent of a general court, unless, within two years after becoming possessed of any ecclesiastical employment, they subscribe.

Every one who has attended to the observations in this and the preceding chapter, must know what judgment to form of these regulations.

Let us suppose that all the clergy in *London* and *Middlesex* came into this association from the first; and that one with another they are subscribers of two guineas annually; and that there are among them as many unmarried persons as married.

In this case, it may be learnt from Quest. XIII, that the annuity to which widows should be entitled, (supposing no allowance to the children of any that die) ought not to exceed 10 or 11 guineas at most; and that, besides the annual subscriptions, there ought to have been a fine paid at the commencement of the scheme, by every married person, of six guineas at least, or, by the whole number of subscribers, three guineas. If the number of married members is double the unmarried, the annuity ought not to exceed eight guineas; and the fine from every member should be about four guineas.—The order, that only a fourth part of the annual subscriptions and interest shall be divided the first three years, half the next four years, and three quarters the next five, is without reason; because the number of claimants, for the first 12 years of the scheme, will be so few, that it will not be possible, during that time, that there should be occasion for dividing any proportions so large of the annual subscriptions and interest, unless they are indeed beyond all bounds too little.—After 12 years, the number of annuitants will go on increasing for near 50 years, as appears from Quest. III. The consequence, therefore, of dividing, after that time, the whole amount of the annual subscriptions and interest, will be a constant yearly diminution in the dividends for near 50 years; and making the payments to the first claimants much more considerable than they ought to be, at the expence of all subsequent claimants—For these reasons, it appears to me out of all doubt, that this scheme is by no means likely to answer the good ends proposed by it; and that, therefore, it will be best to lay it aside. At the time it was settled, it was, I find, further agreed, that the annual subscriptions of the *laity*, together with the interest of their benefactions, unless otherwise directed by the donors; and the annual subscriptions of such of the clergy as shall so direct, shall make a *charitable fund* to be applied to the relief of the distressed widows or children of all the clergy within the limits I have mentioned, whether subscribers or not, provided that in no one year of the first twelve more than £20 be given out of the fund to any one family.—This is an excellent design; and if the money arising from all the subscriptions is thrown into this fund, an important means of relief may be provided for such of the more indigent widows and families as will accept the help of charity.

There is one more scheme of particular consequence, which I must take notice of: I mean, that which is established by Act of Parliament, among the ministers and professors in *Scotland*, for making provision for their widows and orphans. The last mentioned scheme, and also several others of the same kind[k] in this kingdom, have been formed on the model of this: and the success with which it has been hitherto attended, is one of the principal causes to which they have owed their rise. It is, therefore, proper I should give some account of it; and it will be sufficient with this view to mention, "that for an annual payment, which "begun immediately, of *five guineas* from "1011 contributors, 667 of whom are married persons, besides a tax on weddings, "producing about £142 *per annum*, it entitles every widow to an annuity of £20 "during widowhood, and also every family "of children that shall be left by such "members as die without leaving widows, "to £200." This scheme contains a variety of other particulars; but this is its substance—It commenced on the 25th of *March*, 1744; and from that time to the 22d of *November*, 1770[1], or in 26 years and near 8 months, 151 ministers and professors died, and left 151 families of children without widows; that is, 5.66 such families were left annually; and the annual disbursements to them have therefore been £1132. Subtract this sum from £5450, the whole an-

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[k] There is one among the Dissenting Ministers in the counties of *Chester* and *Lancaster*; and another among the Dissenting Ministers in *Cumberland*, *Northumberland*, *Westmorland*,

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Westmorland, and *Durham*.—Even the *London Annuity* Society, though its plan is totally different, professes to form itself on the principles of the *Scotch* establishment, and to derive encouragement from it.

[1] In Nov. 1779, or 35 years and 8 months, 199 ministers and professors had died, and left 199 families of children without widows; that is 5.58 annually.

nual income; and the remainder, or £4318 *per annum* will be the standing provision for bearing the expence of all the annuitants possible to be derived from 667 marriages. Such an annual payment, or 4.27 each from 1011 contributors, is the same with 6.55 each, from 667 contributors; and, consequently, it appears, that in this establishment a contribution is received equivalent to an annual payment beginning immediately, of £6.55 from every married man, in order to entitle his widow to an annuity of £20 during her widowhood.

In the societies mentioned in the last section, annuities increasing from £20 to £40 are promised to widows for an annual payment of only 5 guineas[m]. And, in all the societies for the benefit of widows with which I am acquainted, there is an equal or a greater disproportion between the contributions received, and the annuities promised.—With what strange rashness then has the plan of this establishment been copied? And how absurdly have the societies in this kingdom pleaded it as a precedent which encourages and favours them?—It would be trifling to say more on this subject.

It may be observed that the annual income for the support of this establishment, supposing it to have only the benefit of widows in view, ought to be £7.19 *per ann.* from every marriage, according to Question XIII. p. 46, and £7.44 *per ann.* according to the calculation in Note F. at the end of this volume.

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[m] See page 73.

These determinations exceed the income actually provided.' But the excesses are by no means considerable enough, to afford any certain reason for concluding, that the fund of this establishment will prove insufficient. I was, however, once led to entertain some doubts on this subject. And in these doubts I thought myself confirmed by observing, that, in the calculations[n] made at the commencement of the scheme, the number 333 was stated, as the *maximum* of widows living at one time, likely to come upon it, or to be derived from 20[o] widows left annually; and also, that 40 years was stated as the number of years necessary to bring on this *maximum*; whereas I was satisfied, that the *maximum* of widows would not prove much less than 400; nor the number of years necessary to bring it on, less than 60. In the former editions of this work, I gave a distinct account of this. But I have lately received such information[p] as hath convinced me that my doubts have been in a great measure groundless. I have learnt, in particular, that there have been several calculations subsequent to those I had seen; and that this establishment has enjoyed advantages and provisions for its support which I was unacquainted with, and which give reason for expecting that it will indeed be able to bear the expence of 400 annuitants, should so many come upon it. I should only tire most of my readers, were I to enter into an account of these advantages and provisions. It will be of more importance to take this opportunity to observe, that the probabilities of life from which the determinations I have mentioned are derived, though much lower than the probabilities of life among the ministers and their wives in SCOTLAND[q], are yet such as give the values of reversions depending on survivorships among them too high.

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[n] See Table III. in a book printed at *Edinburgh* in 1748, entitled, Calculations, with the Principles and Data on which they are instituted, relative to a late Act of Parliament, entitled, *An Act for raising and establishing a Fund, for a provision for the widows and children of the ministers of the church, and of the heads, principals, and masters of the universities of* Scotland; shewing the rise and progress of the Fund.

[o] See note A, at the end of this volume. See likewise the note in p. 89.

In order to understand this, it must be considered, that the difference between the probabilities of life in different situations, takes place much more in the first and middle than in the last stages of life; and that the effect of this must be to *increase* the duration of *joint lives*, and at the same time to *lessen* the duration of *survivorship* in those situations which are most favourable to health. Or, in other words, to render the duration of marriage in such situations, greater than it would otherwise be in proportion to the duration of widowhood; and, consequently, to reduce the present value in annual payments during marriage, of any given annuity payable during widowhood. For instance. Were the probabilities of life among the ministers and their wives in SCOTLAND the same that they are by Mr. *De Moivre's* hypothesis, or in the *Breslaw* and *Northampton* Tables of Observation, the duration of marriages among them, taken one with another, could not be more than 19 years. The duration of widowhood would be 22 years, and the *maximum* of widows living at one time derived from 667 marriages constantly kept up, would be considerably more than 400. Were the probabilities of life among them the same that they are in LONDON, the duration of marriage would be still *less*, and the duration of widowhood *greater*, and the *maximum* of widows derived from 667 marriages, could not be less than 500. But the fact is, that the duration of marriage among them is 22 years nearly; and that of widowhood about

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[p] I owe this information to the kind and very obliging candour of the reverend and ingenious Dr. WEBSTER, of *Edinburgh*.

[q] More particular notice will be taken of this at the conclusion of the second Essay.

20 years and a half[r]. And it appears also from accounts taken annually, that the number of widows living at one time, derived from the whole body of ministers and professors, does not exceed 400. It is, therefore, certain that a smaller income must be sufficient for the support of this scheme than would be necessary, according to the probabilities of life in the Tables just mentioned. —And upon the whole; after a careful review of all the circumstances of this establishment in its present state, I am well satisfied that the success with which it has been hitherto attended, is likely to continue; and that it will indeed prove a permanent foundation of that assistance to the *widow* and *fatherless* which is intended by it.—Caution, however, and vigilance, will for some time be necessary. Many more years must pass before it can receive a decisive confirmation from experience. Events have hitherto favoured it more than could have been reasonably expected. They may perhaps hereafter try it; and deviations from probability may arise, which cannot be now foreseen.—But I ought to ask pardon for making these remarks. The venerable mi-

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[r] See a note at the conclusion of the second Essay; and also note (F), at the end of this volume. The *maximum* of widows (supposed 395) divided by the number left annually (or 19$\frac{1}{10}$) gives 20$\frac{3}{5}$, the expectation of widowhood. See page 89, and Note (A) at the end of this volume.

nisters and professors concerned will, I hope, excuse me. They are eminently distinguished by their abilities and knowledge: and can have little need of any information which I am able to give them.

Further Account of the Association *among the* London *and* Middlesex *Clergy; and of the* Establishment *among the* Ministers *and* Professors *in* Scotland.

THE Clergy of London and Middlesex agreed, in 1775, to new rules and orders, by which such fines were required (on account of a subscriber's exceeding the age of 40, and being older than his wife) and such reductions made in the annuities as would probably have rendered the contributions adequate to the expences of the association. But the event has been, that in consequence of this necessary reformation, the Association has dwindled, and is now sunk so low as not to be likely to subsist much longer.

On the contrary. The Establisment among the ministers and professors in Scotland has prospered to a degree which gives reason to believe that it cannot fail to answer the hopes of the venerable body interested in it. This has been owing chiefly to the great ability and faithful zeal of the Rev. Dr. Webster, its founder and conductor.—To the account already given of it in the Second

Section, I will take this opportunity to add the following particulars.

Dr. Webster (having, when the plan was first formed 38 years ago, no certain *data* to go upon) assumed 52 as the medium age at which the widows of ministers would commence annuitants. By calculating on this supposition, and taking the chances of life as they are in Dr. *Halley's* Table, he found that the number of annuitants on the scheme at Lady-day 1780 would be 310. —The fact is, that they were then 304; and that consequently there was, even in this way of calculating, a difference of *six* in favour of the funds which support the scheme. ——Since the establishment of the scheme it has been discovered, that the medium age just mentioned does not probably exceed 47. Dr. Webster, therefore, some years ago, in order to put the scheme to a severer trial, instituted a new calculation, on the supposition that the medium age is no more than 44; and found that on this supposition the number of annuitants at Whitsuntide 1780 would be 328. This has made a still greater difference in favour of the Establishment, and gives a very encouraging prospect of its stability; a sufficient income having been in reality provided for bearing the expences, had the annuitants increased as in this last calculation.

Had 52 been the mean age of the widows when they commence such, the *maximum* of widows living at one time derived from 20 left annually, would be 334, according to Dr. *Halley's* Table; but supposing it no more than 44, this *maximum* would exceed 400; and the enquiries which have been made, give reason to expect that it will not fall much short of this number. Dr. Webster, therefore, has in his last calculations, reckoned upon the increase of the annuitants to this number; and for this reason, and to secure more certainly the Establishment, a new act of parliament was procured in 1779, by which, among other new provisions, it was ordered, that the increase of the capital (then amounting to £75,088) should not be discontinued till it rose to £100,000.——This capital, joined to the annual contributions, will probably be an ample support to the Establishment, should the number of annuitants (which will go on to increase for near forty years more) become at last 400.——Circumspection and caution, however, continue to be necessary, because still unfavourable events may arise, which no human wisdom can foresee.

Having bestowed a good deal of attention on this institution, I cannot take leave of it without congratulating Dr. Webster on his happiness. By being the founder of this scheme, and by the care with which he has watched its progress, and conducted it to its present state of maturity, he has entitled himself to the blessings of many indigent widows and orphans, and made it impossible that he should be ever remembered in the church of Scotland without gratitude and respect.

It is much to be wished that institutions of the same kind could be establised in *England*. Some efforts have been made to this purpose. The reverend and ingenious Mr. *Gandy* of Plymouth, having with much labour and ability prepared a plan of this kind, endeavoured in 1774 and 1775 to get it established in the diocese of *Exeter*. Had he succeeded, the benefits arising from it would have become in a little time very conspicuous, and an example would have been given which would perhaps have been followed in other dioceses. But he did not meet with sufficient encouragement, and the scheme was given up.—Being unwilling that the time and pains which were employed in digesting and calculating the tables for this scheme should be entlrely lost; I have inserted some of the principal of them among the Tables in the next volume.——The Rev. Mr. *Grant*, of *Henly* upon *Thames*, has been also lately engaged in soliciting encouragement to a similar scheme; and I heartily wish him the success his benevolence and abilities deserve[a].

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[a] This scheme has shared the fate of Mr. Gandy's. M.

Account of a Scheme established among the East-India Commanders.

THE *East-India Commanders* six years ago entered into an association for the purpose of providing for their *nominees*, and did me the honour to desire I would recommend a scheme to them. They approved the following, and have adopted it.—Every member is entitled to £500 payable at his death to his *nominee*, in consideration of £50 at his admission, and *eleven annual payments* of £25, the first to be made at the beginning of the second year, and the right to every payment to depend on the continuance of the life of the subscriber. No subscribers must be admitted whose ages are not less than 50; and their mean ages are reckoned at 40.

In calculating this scheme, interest was reckoned at 3½ *per cent.*[t]; but a much higher improvement has for some years been made of money, and the scheme has escaped the danger of being too much loaded in its infancy. I am satisfied, therefore, that, without

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[t] The value, by Table XVII. in the next volume, reckoning interest at 3½ *per cent.* of an annuity for eleven years, on a life aged 40, the first payment to be made at the end of a year, is by Quest. VI. Chap. I. 7.895.—The annuity, therefore, being £25, its value is £197, to which £50, (the first payment) added makes £247; which is also, by Quest. X. the value, reckoning the same interest, of £500 payable at the death of a person aged 40.—The same contributions, supposing money improved at 5 *per cent.* would entitle a nominee to £600.

out altering the contributions, the sum payable to *nominees* may with perfect safety be increased to £550. Indeed, I should be in no pain were it even advanced to £600, provided only the contributions were made in *guineas*, instead of *pounds*, and all savings made to accumulate in the short annuity for 27 years from Chrismas last.——The progress of such an association will be as follows.—Suppose it to consist of 46 members, kept up from year to year by admitting, as old members die off, new ones at the mean age of 40. At first, according to Mr. *De Moivre's* hypothesis, only *one* member will die annually; but, after a certain period, *two* will die annually. During this interval there will be savings which will raise a stock, the interest of which, when added to the annual contributions, will be just sufficient, when two members come to die annually, to pay two claims.—Supposing each claim £550, the expence of the association, *when greatest*, will be £1100 *per ann.* The contributions at that period will be first, £50 each (or £100 in all) from *two* new members admitted every year; and £25 each (or £450 in all) from 18[u] other members who had not been admitted more than eleven years. The remainder (or £550) necessary to make up £1100 *per ann.* will be the interest of the capital, which, therefore, if lodged in the *three per cents.* must be £18,333. When, therefore, any such association consisting of 46 members has raised this stock; or, if it consists of any other number of members, when it has raised a stock in the same proportion to it, that the standing number of members bears to 46, it will become a reasonable object of consideration whether the increase of its stock should not be discontinued, and all subsequent savings, should any arise, be employed in either lessening the contributions or increasing the claims[x]. SECT.

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[u] Out of a body of new members derived from two admitted every year at 40 for 11 years, it may be expected that two will die before the end of the 11th year.

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[x] The examination of this Society's affairs was referred to me in the year 1797, when it appeared that the number of those members who had compleated their eleven payments amounted to 29, and the number of those who having been admitted at late periods had *not* completed their payments amounted to 17.—At that period, or 21 years after their establishment, 38 deaths had taken place, which exceeded the number that should have happened according to the *Northampton* Table in the proportion of 4 to 3. But the premiums having been computed at 3½ *per cent.* while their money had been improved at 5 *per cent.* the increased number of claims was balanced by the increased accumulation of the capital, and therefore had their stock, which then consisted of £690 *per annum* in the Long Annuities, and an Exchequer Bill for £1000, been sold for the sum which was originally paid for it, a considerable *surplus*, after discharging all demands, would have been left in favour of the Society: But the subsequent depression of the funds so far lessened the value of the Society's capital as, not only to destroy that surplus, but even to produce a *deficiency* to a larger amount. Happily, however, the funds since that period so far recovered as to restore the affairs of the Society, and to render them secure. (*a*)

(*a*) Since writing the above, the purposes for which they

SECT. III.

Of the best Schemes for providing Annuities for Widows.

INSTITUTIONS for providing widows with annuities would, without doubt, be extremely useful, could such be contrived as would be *durable*, and at the same time *easy* and *encouraging*. The nature of things do not admit of this in the degree that is commonly imagined. The calculations and rules, in the preceding chapter, will enable any one to determine in all cases to what reversionary annuities any given payments entitle, according to any given valuation of lives, or rate of interest. From Quest. VII. and VIII. in particular, it may be inferred that (interest being at 4 *per cent.* and the probabilities of life as in the *Breslaw*, *Norwich*, and *Northampton* Tables) for an annual payment beginning immediately of *four* guineas during marriage; and also for a guinea and a half in hand, on account of each year that the age of the husband exceeds the age of the wife; every married man, under 40, might be entitled to an annuity for his widow, during *life*, of £5 if he lives a year; £10 if he lives *three* years; and £20 if he lives *seven* years.

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they were originally associated no longer existing, the Society dissolved themselves, and every Member received the *full* value of his Interest in the Capital. M.

If such a society chuses, that those who shall happen to continue members the longest time, shall be entitled to still greater annuities, six guineas, additional to all the other payments at admission, would be the full payment for an annuity of £25, and 12 guineas for an annuity of £30 if a member should live 15 years.

All bachelors and widowers might be encouraged to join such a society, by admitting them on the following terms.—*Four guineas* to be paid on admission, and *three guineas* every year afterwards, during celibacy; and on marriage, the same payments with those made by persons admitted after marriage; in consideration of which, £1 *per annum*, for every single payment before marriage, might be added to the annuities to which such members would have been otherwise entitled.

For example. If they have been members four years, or made five payments before marriage, instead of being entitled to life-annuities for their widows of only £5, £10, £20, £25, and £30, on the conditions I have specified, they might be entitled to annuities of £10, £15, £25, £30, and £35. Or, if they have been members nine years, and made 10 payments, they might, instead of the same annuities, be entitled to annuities of £15, £20, £30, £35, and £40.—In this case, the contributions of such members as should happen to desert, or die in celibacy, would be so much profit to the society,

ciety, tending to give it more strength and security.

This is one of the best schemes that I can think of, or would chuse to recommend. But in the following scheme there is a simplicity and fairness which seem to give it a particular preference.

Every husband, be his age what it will, for a single payment at admission, of £15, with £1...10*s*. added for *every* year that his age exceeds his wife's, and an annual payment of £5 during marriage, (the first to be made at the end of a year) might entitle his wife, should he leave her a widow, to an annuity of £10 for her life, if he lives *one* year; £11 if he lives *two* years; £12 if he lives *three* years; and so on; the annuity to increase continually at the rate of £1 for every year that the husband lives beyond one year.——Any addition to these payments might entitle to a proportionable addition to the annuity, and to its increase. ——And should any husband under 40 wish to secure a sum for his children, *provided* he should leave no widow, he might for every annual payment of nine shillings, *during life*, entitle them to £50, payable among them at his death, whenever that shall happen. Making all these payments *guineas* instead of *pounds*, might probably be sufficient, if the number of subscribers is considerable, to defray the expences of management.

There is one particular advantage which societies formed on plans of this kind would enjoy [y].—Persons who know themselves subject to disorders which are likely to render them short-lived, will have no great temptations to endeavour to gain admission into such societies; and, if admitted, the danger from them will be less than on any other plan. Were it not for this danger, the following plan might be recommended.

In the plans hitherto mentioned it is implied, that, if either a member or his wife dies within any of the periods specified, the additional annuities, that would otherwise have become due, will be lost. But it would be much more agreeable to a purchaser, that they should be made certain to his wife, provided she lives to the end of these periods, though in the mean time his own life should fail. The value of such annuities may be computed by the rule in Question IX.

Suppose, for instance, the *scheme* to be "that a wife shall be entitled certainly to a "life-annuity of £20, the first payment of "which shall be made at the end of 12 years, "provided she should be then alive, and her "husband dead; or at the end of any year "beyond this term in which she may happen to be left a widow." Suppose it also stipulated, "that she shall be entitled to

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[y] See another advantage mentioned under Question VIII.

"£10 more, or £30 in all, on the same "terms, provided she should live 16 years." —The value of such an expectation (interest being at 3 *per cent*. and the probabilities of life as in the *Northampton* Table of Observations) will be, in the most convenient round sums, supposing none admitted above 50 years of age, seven guineas in annual payments to be continued during marriage, and to begin immediately; besides four guineas in present money for every year, as far as 15 years, that the husband's age exceeds the wife's, if he is between 40 and 50, and three guineas on the same account if he is under 40; Or, if the whole value of the expectation is given in one present payment, £70 added to a guinea and half for every year that the husband's age falls short of 50, besides the payment just mentioned on account of disparity of age.

The value of this expectation at 4 *per cent.* is six guineas in annual payments, besides three guineas in present money, for every year that the husband's age exceeds the wife's, if he is between 40 and 50; and two guineas, if he is under 40: Or, if the whole value of the expectation is given in one present payment, £56 added to £1...5*s*. for every year that his age falls short of 50, besides the payment last mentioned on account of inequality of age[z].

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[z] Supposing 16 years the only term, the annuity £20, and interest at 4 *per cent.* the proper payments will be nearly, in the case of equal ages and *single* payments, £46

He that will give himself the trouble to calculate, agreeably to the directions in the Questions to which I have referred, will find that, taking all particular cases together, the rules now given come as near the truth as there is reason to desire in an affair of this nature, the *defects* in some cases being nearly compensated by the *excesses* in others.

These determinations are agreeable to the probabilities of living in Dr. *Halley's*, as well as the *Northampton* Table of Observations; or Tables 5th and 17th in the next volume. These Tables seem to give a proper *medium* between the different values of *town* and *country* lives. In the country the probabilities of living are much higher; but in *London*, and probably in all *great* towns and some *smaller* ones, they are much lower.

It is proper to add, that, according to the values of lives deduced both from the *London* and Dr. *Halley's* Table, and taking interest as low as 3 *per cent.* all women

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£46—£40—£29, as the age of the man is 30, 40, or 50. Or, in *annual* payments, £3.80—£3.66—£3.13.—Supposing the woman's age 10 years less than the man's, the same values will be, in *single* payments, £58.92—£56.56—£53.66—in *annual* payments £4.63—£5—£5.41. It appears, therefore, that a society, supposing money improved at the rate of 4 *per cent.* might entitle all married men *indiscriminately*, who are under 50 years of age, to such an expectation as this for their wives, for either £60 in *one* payment, or five guineas in *annual* payments.—But equity requires, that different payments should be made according to the different comparative ages of men and their wives.

whose husbands are under 50 years of age, might be entitled to an annuity of £24 during *life* (the first payment to be made at the end of the year in which they shall be left widows) for the sum of £100, supposing £3 additional given on account of every year that they are younger than their husbands.—At 4 *per cent.* an annuity of £30 might be granted on the same terms.

In the year 1690, the company of *Mercers* in *London* adopted such a scheme as that last mentioned. For £100 in *one present payment,* they entitled every subscriber to a *life-annuity* for his widow of £30; and this, at that time, (when money bore 8 *per cent.* interest) was considerably less than the value of the money advanced, supposing men and their wives of equal ages. As the interest of money sunk, they sunk also the annuity, first to £25 and then to £20 and £15. But at last, after carrying on the scheme for above 50 years, finding the burden of the annuitants too heavy, and likely to go on increasing, they were obliged to drop the scheme and to stop payment. In a little time, however, by a parliamentary aid of £3000 *per annum,* they were restored to a capacity of making good all their engagements, and of paying their arrears.—Their failure is, indeed, much to be lamented; for, in consequence of it, the public has lost the benefit of an institution, that for many years promised the happiest effects, by encouraging marriage, and affording relief to indigence. The rapid fall of the interest of money; their admitting purchasers at too advanced ages; and, particularly, their paying no regard to the difference of age between husbands and their wives, must have contributed much to hurt them. Some of the principal causes, therefore, which have rendered them unsuccessful, may be now avoided.

It must, however, be remembered, that the issue of the best schemes of this kind must be in some degree uncertain. For want of proper observations[a], it is not possible to determine what allowances ought to be made, on account of the higher probabilities of life among females than males. No prudence can prevent all losses in the improvement of money; nor can any care guard against the inconveniences to such schemes, which must arise from those persons being most ready to fly to them who, by reason of concealed disorders, feel themselves most likely to want the benefit they offer.

The societies, therefore, on which I have remarked in the first section of this chapter, would have reason to take warning from what has happened to the *Mercers' Company*, were the schemes on which they are formed perfectly unexceptionable. But I have demonstrated that these schemes are very defective; and that the longer they are carried on, the more mischief they must produce. 'Tis vain (as appears from Quest. III.) to form such establishments with the expectation of seeing their fate determined soon by experience. If not more extravagant than any ignorance can well make them, they *will* go on prosperously for 20 or 30 years; and, if at all tolerable, they *may* support themselves for 40 or 50 years; and at last end in distress and ruin. No experiments, therefore, of this sort should be tried hastily. An unsuccessful experiment must be productive of very pernicious effects. All inadequate schemes lay the foundation of *present* relief on *future* calamity, and afford assistance to a *few* by disappointing and oppressing *multitudes*.

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[a] This defect will, I hope, be in some measure removed by the Observations and Tables in the next volume.

As the persons who conduct these schemes can mean nothing but the advantage of the public, they ought to listen to these observations. At present their plans are capable of being reformed; but they cannot continue so always; for the greater number of exorbitant payments they now make to annuitants, the more they consume the property of future annuitants, and the less practicable a retreat is rendered to a rational and equitable and permanent plan[b]. They should, therefore, *immediately*[c] either reduce their schemes, or change them into one of those which I have proposed. But, I am afraid, this is not to be expected. The neglect with which they have received some remonstrances that have been already made to them, gives reason to fear, that what has been now said will be in vain; and that those who are to come after them, must be left to *rue* the consequences of their mistakes.

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[b] See p. 90, 91. Sect. I.

[c] Thus; were the *London* Annuity Society to make their

SECT. IV.

Account of some foreign Institutions for the Benefit of Widows.

IN the Preface to the first edition of this Treatise, I took notice of an institution for the sale of annuities payable on survivorship, established at *Amsterdam*, which seemed to be then much encouraged, and into which, I had been informed, many had entered from different parts of *Europe*. This was so wretched a deception that it was impossible it should long stand its ground; and I am told that it now exists no more. I have, therefore, expunged the notice I took

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their lowest annuity £10, the next £20, and the highest £30, they would probably be safe. But, after proceeding on their present plan some years longer, such a reduction would by no means be sufficient.

of it in that Preface; and I will not here give any further account of it.

In 1739, an institution was established in *Denmark* under the patronage and guaranteeship of the King of Denmark, which, without regarding ages, promised pensions to widows at the rate of 40 rixdollars *per ann.* for life, from the commencement of widowhood, for every present payment of 110 rixdollars. This being less than the true value of such pensions, the fate of this scheme has been the same with that of the *Mercers'* Company mentioned in page 128. At the end of the year 1778, its whole fund was exhausted, and the King of Denmark found himself burdened with the support of 700 widows, and an obligation to support as many more as would be derived from 1500 marriages then remaining undissolved.

At *Bremen*, an institution was established in 1760, which promised annuities to widows for a payment on admission of a sum equal to one yearly payment of the annuity purchased, and an annual contribution during marriage of 15 *per cent.* (or a little more than a 7th) of the annuity. These payments are not much more than half the proper compensation for the annuities. The conductors of the scheme have therefore been obliged to reduce the annuities 10 *per cent.*; and they will soon be obliged to reduce them much more.

The states of the duchy of *Calenberg*, of which *Hanover* is the capital, established in 1767

1767 a like scheme, but on terms still more deficient; for, though it differed from the two former schemes in paying a regard to the ages of married persons, yet, notwithstanding several augmentations, the contributions required by it did not two years ago (*) come up to half the value of the annuities. Great numbers, influenced probably by the lowness of the terms and the authority of the states, have been induced to encourage this institution. In 1779, it had annuities to pay to 600 widows, and consisted of no less than 3800 members or subscribers whose widows would be entitled to annuities. In consequence of a rapid increase, its insufficiency was not then become palpable enough to force either a dissolution, or a timely and effectual reformation. It was, therefore, likely to lay the foundation of great confusion and distress.

There are probably many other foreign bubbles of this kind, of which I have no knowledge. The information which has enabled me to give this account I owe to Mr. OEDER of *Oldenburg*; and it is with particular satisfaction, that I can from him on this occasion add an account of one foreign institution for the benefit of widows which is founded on just principles, and likely to be productive of great good.—The plan of this institution has been formed and the calculations for settling its terms have

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(*) That is, about the year 1788.—M.

been made by Mr. *Oeder*, who appears indeed to possess an acquaintance with this subject so extensive and correct as to be perfectly qualified for such an office.——This institution is intended only for the benefit of the inhabitants of the diocese of *Lubec* and the duchy of *Oldenburg*; and the sovereign of this state has himself given it the sanction of a statute, and guaranteed to his subjects the advantages it promises. At prices deduced by calculations at 4 *per cent.* from Mr. *Susmilch's* Table of Observations, (See the Tables in the next volume), and agreeing nearly with the prices deduced by the rules in Quest. I. and IV. from the *Northampton* Table (or Table XVII. in the next volume), it offers to a married man any annuity for his widow not exceeding 500 rixdollars, (or about £88 *per ann.*) [d] payable for life; but with a power reserved to the husband of directing that it shall be applied to the support of his children in case his widow should marry.

This institution farther enables a parent to provide for his children annuities, (not exceeding 500 rixdollars) payable to them in the event of their survivorship, till they are 25 years of age.

For example. To a husband, aged 35, this institution promises a life annuity of £10 payable to his widow, for either an *annual* payment of £2...15*s.* £3...1*s.* £3...7*s*...4*d.* &c. or a *single present* payment of £31...7*s.* 3*d.* £36...6*s.* £41.. 6*s.* &c. according as he is of the *same* age with his wife, or 5, 10, &c. years *older.*

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[d] Six German rixdollars make about a guinea.

And if he wishes to make a provision for any of his children, provided he should leave them orphans under age, he may purchase annuities payable to them from the time they shall happen to survive till they are 25 years of age, at the rate of an annuity of £10 for every *annual* payment during the joint lives, of £1...17*s*...4*d.* £1...14*s.* £1...7*s*...4*d.* or a *single* payment of £19... 14*s.*—£18...12*s*....9*d.* £12...7*s.* according as the child's age is two, five, or ten.[*] years. These values are greater or less as the age of the parent is greater or less; and all the prices of such annuities, and also of annuities for widows, are specified in Tables, for all ages and all *differences* of ages.

The sufficiency of the receipts to answer the expences in this institution, as far as it provides annuities for widows, has been proved by Mr. *Oeder* in the clearest manner from accounts which have been collected in the duchy of *Oldenburgh* of the duration of 1273 marriages, and compared with accounts of the duration of the widowhoods derived from these marriages.——One circumstance in these accounts deserves particular notice.

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[*] It is obvious that the values in this case for every annuity of £1 are, in a *single payment*, the *excess* of the value of the life of the child for as many years as his age is less than 25, above the value of the joint lives for the same time found by the *Scholium* to Quest. VI.; and, in *annual payments* beginning immediately, the *Quotient* arising from dividing the single payment by the value just mentioned of the joint lives, with unity added.

The ages of the men, one with another, when the marriages just mentioned commenced, was $32\frac{1}{2}$ years; of the women 28 years. The men lived after marrying $27\frac{1}{4}$ years; the women, 31 years and nine tenths. The former, according to Mr. *Susmilch's* Table (and also nearly according to the *Northampton* Table of Observations), should have lived only 27 years and one tenth; and the latter, 29 years and eight tenths. The former, therefore, having exceeded the duration of life exhibited in the Tables only *six tenths of a year*, but the latter having exceeded it above *two years*, it follows that women, notwithstanding the hazards of the critical periods and of child-bearing, live longer than men.

I will add, that by examining 154 of these marriages, I find their duration to have been, one with another, 21 years and a quarter, and the duration of the survivorship of the widows derived from them, 19 years. Had Mr. *De Moivre's* hypothesis of an equal decrement of life been just, the latter would have been longer than the former. The reason why the contrary happens has been given in p. 113, &c.

In these marriages (if I may judge from

examining only 140 of them) *four* widows were left to *three* widowers, which shews a chance of survivorship in favour of the wife in marriage, greater than could have taken place, had there been no other reason for it than inferiority of age.

At *Hamburgh*, an annuity scheme has been lately established of a more comprehensive nature than any of the schemes which have been hitherto mentioned; but the account of it will be more properly given at the conclusion of the last section of this Chapter.

SECT. V.

Of Schemes for providing Annuities for old Age.

A GENERAL disposition has lately shewn itself, to encourage schemes for granting *annuities* to persons in the latter stages of life; and this has occasioned the 6th Question in the former chapter; and, as a further and more particular direction in cases of this kind, I have thought it necessary here to give the following Table.

Values of 1*l* *per ann.* for life, after 50, to persons whose ages are	Values in one *present payment*, interest 4 *per cent.*	Interest 3 *per cent.*	Values in *annual payments* till 50, to begin at the end of a year, interest 4 *per ct.*	Interest 3 *per cent.*
10	1.235	2.015	.079	.113
15	1.583	2.444	.106	.146
20	2.028	2.089	.146	.193
25	2.594	3.644	.203	.259
30	3.369	4.508	.297	.366
35	4.446	5.667	.466	.559
40	5.953	7.232	.822	.950
Values of the same annuity after 55, to ages			Values in annual payments till 55.	
30	2.114	2.937	.197	.211
35	2.722	3.632	.241	.297
40	3.732	4.708	.394	.464
45	5.088	6.115	.703	.803
Values of the same annuity, after 60, to ages			Values in annual payments till 60	
35	1.667	2.290	.135	.168
40	2.234	2.923	.203	.245
45	3.043	3.811	.327	.384
50	4.255	5.061	.600	.679

The numbers in the 2d and 3d columns of this Table, multiplied by any annuity, will give the value of that annuity in a *single* payment, to be enjoyed for life, by the ages corresponding to those numbers in the first column, *after* the age at the head of that column,

column.—And in the same manner; the numbers in the 4th and 5th columns will give the values in *annual* payments.—Thus: The value of £44 *per annum*, to be enjoyed for life, after 50, by a person now 40, (interest at 4 *per cent.*) is 5.953 multiplied by 44, or £261.9 in a *single* payment; and .822, multiplied by 44, or £36.17, in *annual* payments till 50, the first payment to be made at the end of a year.

In order to find the same values, partly in *annual payments*, and partly in any given *entrance* or *admission money*; say, "As the va-" "lue of the *given annuity* in a *single* payment, "(found in the way just mentioned) is to the "given *entrance money*; so is its value in *an-* "*nual* payments, to a fourth proportional; "which, subtracted from the value in *annual* "*payments*, the *remainder* will be the annual "payment due, over and above the given "entrance money."

EXAMPLE.

Suppose a person now 40, to be willing to pay £200 entrance-money, *besides* such an annual payment for 10 years as shall, together with his entrance-money, be sufficient to entitle him to a life-annuity of £44 after 50. What ought the annual payment to be?

ANSWER.

£8.55.—For, £261.9. is to £200 as

£36.17, to £27.62; which, subtracted from £36.17, the remainder is £8.55.

This Table has been calculated from the *probabilities* of living in Table V. in the next volume, and Mr. *De Moivre's* valuation of lives. The probabilities of life among the inhabitants of *London*, are (as I have often had occasion to observe) much lower than among the generality of mankind; and the values in the preceding Table, had they been given agreeably to the *London* Observations, would have been less. But, certainly, an office or society, that means to be a permanent advantage to the public, ought always to take higher rather than lower values, for the sake of rendering itself more secure, and gaining some *profits* to balance *losses* and *expences*.

There have lately been established, in *London*, several societies for granting such annuities as those now mentioned; and he that will compare their true values, as they may be learnt from the preceding Table, with the *terms* of admission into these societies, as given in their printed *Abstracts* and *Tables*, must be surprised and shocked. They are all impositions on the public, proceeding from ignorance, and encouraged by credulity and folly.

It has been shewn; that the proper payment (allowing compound interest at 4 *per cent.*) for an annuity of £44 to be enjoyed by a person now 40, for what may happen to remain of his life after 50, is £200 in *admission-money;* besides £8.55. or £8...11*s.* in annual payments till he attains 50, the first of these payments to be made at the end of a year.—The conditions of obtaining this annuity, according to the present Tables of the *Laudable Society of Annuitants for the Benefit of Age,* (*) consisting of about 1300 members are [f] £76...17*s.* in *admission-money*; and £6...14*s.* in *annual payments.*—According to the Tables of the society of *London Annuitants for the Benefit of Age,* the conditions of obtaining the same annuity are £30 in *admission-money,* and £10 in *annual payments.*—The *Equitable* Society of Annuitants requires for the same annuity £38...10*s.* in *admission-money,* and £13 in *annual payments.* The true value is, over and above the *admission*-money just mentioned, an *annual payment* of £30...17*s.* (interest reckoned at 4 *per cent.*) or an *annual payment* of £36...15*s.* (interest reckoned at 3 *per cent.*)—The *London Union Society for the comfortable support of aged members* promises an annuity of no less than 50 guineas for life, after 50, to a person now 40, for £40...10*s.* admission-money, and £7 in annual payments.

The *Amicable Society of Annuitants for the*

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[f] The first members of this Society have paid no admission-money; and are now expecting £44 *per annum* for contributions which do not entitle them, one with another, to £10 *per annum.*

(*) Most of these Societies have long ago ended in ruin.—M.

benefit of age, promises an annuity of £26 *per annum*, for life, to a person now 40, after attaining 50, for £28...16*s*. in *admission-money*, and £6 in *annual payments*.—The true value of this annuity is £28...16*s*. in *admission-money*, and £17...8*s*. in *annual payments*, (interest supposed at 4 *per cent.*); or the same sum in *admission-money*, and £20...18*s*. in *annual payments*, interest supposed at 3 *per cent*.

The *Provident Society for the benefit of age*, consisting of 1280 members, promises an annuity of £25 to a person now 40, after the age of 50, for 34 guineas in admission-money, and eight guineas in annual payments. The true value is, 34 guineas in *admission-money*, and £15...12*s*. in *annual* payments, interest at 4 *per cent.*; or, the same sum in *admission-money*, and £19 in *annual* payments, interest being at 3 *per cent*[g].

But I will not tire the reader, by going, in this manner, through the schemes of all these societies. The contrivers of them, it is certain, can know nothing of the principles on which the rule in Quest. VI. and the demonstration of it in Note (B) at the end of this volume, are founded; and, therefore, if unwilling to be guided by the authority of mathematicians, it may not be possible to convince them of their mistakes. I will, however, offer to them the following demonstration, which will be understood without difficulty, by every one who knows how to compute the increase of money at compound interest.

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[g] The account here given of the terms on which a person whose age is 40, is admitted into these societies, I have taken from their printed Tables as they stood at the end of the year 1770.—In the younger ages, the deficiencies are greater.

The value of a life at 50, (interest being at 4 *per cent.*) is 11$\frac{1}{3}$ years purchase by Table VI, in the next volume. For an annuity, therefore, of £44 *per annum* for life, to be enjoyed by a person at this age, £498 ought to be given. *Two* in *three* of a number of persons at the age of 32 will (by Tables V, VIII, and XVII, in the next volume) live to 50; and therefore, in order to be able to pay an annuity to them of £44 for life, after 50, the money now advanced by every *three*, ought to be such as will, in consequence of being laid up to be improved, increase in 18 years to double £498, that is, to £996.—From the preceding Table it may be learnt, that the money which ought to be advanced by every single person is £165, or by *three* persons £495; and this, in 18 years, will (as may be learnt from Table III. in the next volume) double inself, or increase to just the sum that will then be the value of the annuities to be paid.—But the money required in this case by the *Laudable Society*, is £14...11*s*...9*d*. from each member at admission, besides an *annual* payment of £4. The admission-money, therefore, of two members, being £29...3*s*....6*d*. may be increased to twice this sum, or to £58...7*s*.

An annual payment of £4 for 18 years will, if perfectly improved at 4 *per cent.* compound interest, increase to £102; and two such annual payments will increase to £204, as may be learnt from Table IV. in the next volume.

The whole pay, therefore, of *two* members will produce at the end of 18 years £262...7*s.*—A third part, I have said, will die before 50 years of age, and these will live one with another 9 years. An annuity of £4 for this time, will produce a capital of £42...6*s.* See Table IV. in the next volume; and this capital improved for nine years more will increase to £60. The whole profit, therefore, from the member who will die is, his admission-money doubled and added to £60, or £89...3*s.*...6*d.* And this sum added to £262...7*s.* makes £351...10*s.*...6*d* the *whole* money with which the society can be provided, at the end of 18 years, to bear the expence of *two* life-annuities, worth together £996.

By a similar computation it may be found, that the improvement of money at only 3 *per cent.* will *sink* the *former* sum to £324 at the same time that the values of the *annuities* will be *raised* to £1100.

The deficiencies in the schemes of most of the other societies, are no less considerable.—What confusion then must they produce some time or other? How barbarous is it thus to draw money from the public by promises of advantages that *cannot* be obtained? Have we not already suffered too much by *bubbles?*

I have said, that these societies are "impositions on the public, proceeding from "ignorance, and supported by credulity and "folly." But this is too gentle a censure. There is reason to believe that worse principles have contributed to their rise and support. The present members, consisting chiefly of persons in the more advanced ages, who have been admitted on the easiest terms, believe that the schemes they are supporting will last *their* time, and that *they* will be gainers. And as to the injury that may be done to their *successors*, or to *younger members*, it is at a distance, and they care little about it. Agreeably to this principle, the founders of these societies begin so low as not to require perhaps a *fourth* or a *fifth* of the values of the annuities they promise. Afterwards they advance gradually, just as if they imagined, that the value of the annuities was nothing determinate, but increased with every increase of the society. But, as no ignorance can believe this, the true design appears to be, to form soon as large a society as possible, by leading the unwary to endeavour to be *foremost* in their applications, lest the advantage of getting in, on the easiest terms, should be lost. It is well known, that these arts have succeeded wonderfully; and that, in consequence of them, these societies now consist of persons who, for the *same* annuities, make higher or lower payments according to the time when they have been admitted; and the generality of whom, therefore, must know, that either more than the values have been required of the members last admitted; or, if not, that they are themselves expecting considerable annuities, for which they have given no valuable consideration, and which, if paid them, must be *stolen* from the pockets of some of their fellow-members[h].

I do not, however, mean to condemn all institutions of this kind. They may be very useful, if the full values are taken, and proper care is used in the *improvement* of money. Interest, in these cases, ought not to be reckoned higher than 3 *per cent.* and, supposing money improved at this rate, a person, for a single payment of £50 before he is 40, might be entitled to a life-annuity of 10 guineas *after* 55; or, if he chuses it, to a life-annuity of £17 after 60. But if he pays the same sum before he is 34, he might be entitled to a life-annuity of £14 after 55, or £22 *after* 60. £25 would purchase for him *half* these annuities, and £100 *double*.

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[h] If any person wants more information than I have given him concerning these societies, he should consult a work of great merit, published since the second edition of this treatise, and entitled, Calculations *deduced from first Principles, in the most familiar Manner, by plain Arithmetic, for the Use of the Societies instituted for the Benefit of old Age; intended as an Introduction to the Study of the Doctrine of Annuities.* By Mr. *Dale*, a Member of one of the Societies. In this Treatise there is not only a very ample account given of the insufficiency and iniquity of the schemes of these Societies, but the principles on which the values of all annuities on single lives are determined, and the method of calculating them, are explained with the greatest clearness.

A society or office that would go on this plan, might do great service. Persons in the lower stations of life might be brought to a habit of industry, in the beginning of life, by striving to get £25, or £50, beforehand in order to purchase such annuities, and thus to make provision for themselves in the more advanced parts of life, when they will be incapable of labour[1].

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[1] The benevolent Mr. *Howard*, in his *State* of the Prisons in *England* and *Wales*, p. 60. Octavo Edit. gives the following account of an institution at *Harlem* in *Holland*. "In this city," he says, "there is a noble hospital, airy and spacious, called the *Proeveniers*, in "which the persons admitted are decently provided with "meat, drink, and lodging, during their lives, and a "burying-place at their decease. Persons of all ages, "from 20 to the most advanced periods, are admitted, "though it is not common for any under 40 to apply for "admission. At entrance each person pays a greater or "smaller sum, according to his age. If his age is 30, "he pays 4500; if 40, he pays 3900; if 50, he pays "3300; if 60, 70, &c. he pays 2700, 2100, &c. florins; "and in proportion to these sums, at all the intermediate "ages.—A common table is provided for all that are not "sick or infirm.—It often happens that there are not vacant places when persons desire admission. But many "secure places before a vacancy, by paying 200 florins "in advance of their entrance-money, in consequence "of which their names are inscribed in a list, and the

"money

It is proper to observe here, that institutions of this kind would furnish one of the *safest* ways of providing for widows.—A married man might, by paying £100 before his wife attained 40, intitle her, after 55, or 60, to a life-annuity of £21, or £34. Or, by paying the same sum before she attained 34, he might intitle her, after the same ages, to a life-annuity of £28, or £44[k]; and in this case he would have a chance of sharing himself in the benefit of the annuity.

I have called this the safest way of providing for widows, because attended with none of the dangers arising from disproportion of age between men and their wives, and from the admission of persons labouring under concealed distempers.

I cannot conclude this Section, without mentioning the following plan of a provision for Old Age.

Let 13 guineas be given as *entrance-money*; and let besides £1, £2, £3, £4, &c. be given at the beginning of the 1st, 2d, 3d, 4th, &c years, as the payments for these years respectively; and let the last payment be £16 at the beginning of the 16th year. All these payments put together will, according to the

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“ money deducted from the sum required when they
“ enter.—Those who chuse to leave the house, receive,
“ during the remainder of their lives, a certain but small
“ interest for the sum which they paid at admission.”

[k] The same payment before 30, would entitle to an annuity of £22 after 50.

probabilities of life in the 5th or 8th Tables in the next volume, (interest being at 4 *per cent.*) entitle a person, whose age was 40 when he begun them, to an annuity, after 15 years, beginning with £15, and increasing at the rate of £1 every year, till at the end of 15 years, or[l] when he has reached to 70, it becomes a standing annuity of £30 for the remainder of his life.

If the addition of three guineas is made to the *entrance-money*, for every year that any life between 30 and 40 falls short of 40, the value will be obtained nearly, of the same annuity to be enjoyed by that life, after the same number of years, and increasing in the same manner, till it becomes *stationary* and *double.*——This plan is particularly inviting, as it makes the *largest* payments become due, when the *near* approach of the annuity renders the encouragement to them *greatest;* and as, likewise, the annuity is to increase continually with age, till it comes to be highest[m], when life is most in the decline,

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[l] According to the probabilities of life in the *London* Table, this annuity should be greater.—A *Theorem* for finding what the annuity ought to be in these cases, is given in Note (I), at the end of this volume.

[m] The lower part of mankind are objects of particular compassion, when rendered incapable, by accident, sickness, or age, of earning their subsistence. This has given rise to many very useful societies among them, for granting relief to one another, out of little funds supplied by

cline, and when therefore it will be most useful.—It is further a recommendation of this

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by *weekly* contributions. A society of this kind, formed on the following plan, would probably thrive.

Let the society, at its first establishment, consist of 100 persons, all between 30 and 40; and whose mean age may therefore be reckoned 36; and let it be supposed to be always kept up to this number, by the admission of new members, between the ages of 30 and 40, as old members die off. Let the contribution of each member be four-pence *per* week, making, from the whole body, an annual contribution of £85 . 17*s*.—Let it be further supposed, that seven of them will fall every year into disorders, that shall incapacitate them for seven weeks.—£30...12*s*. of the annual contribution will be just sufficient to enable the society to grant to each of these 12*s*. *per* week, during their illnesses. And the remaining £55 *per annum*, laid up and *carefully* improved at 3½ *per cent*. will increase to a capital that shall be sufficient, according to the chances of life in Tables V, VIII, and XVII, in the next volume, to enable the society to pay to every member, *after* 67 years of age, or *upon* entering his 68th year, an annuity, beginning with £5, and increasing at the rate of £1 every year for seven years, till, at the age of 75, it came to be a standing annuity of £12 for the remainder of life.

Were such a society to make its contribution *seven pence per* week, an allowance of 15*s*. might be made, on the same suppositions, to every member during sickness; besides the payment of an annuity beginning with £5 when a member entered his 64th year, and encreasing for 15 years, till, at 79, it became fixed for the remainder of life at £20.

If the probabilities of life are lower among the labouring poor, than among the generality of mankind, this plan will be so much the more sure of succeeding.

In 1773, a pamphlet was published, entitled, *A Proposal for establishing Life Annuities in Parishes, for the Benefit* this plan that less depends in it on the *improvement* of money than in most other plans.

Benefit of the industrious Poor."—"It is," says this writer, "a common (*a*) observation that the money annually "raised for the poor, amounts to, *at least*, a million a "year; and that yet in many places they are but indifferently provided for. To make provision for one's "old age is so natural a piece of prudence, that it seems "at first sight wonderful, that it should not be generally "practised by the labouring poor, as it is almost universally by persons in the higher paths of industry: "Nor can their negligence in this respect be accounted "for, in any other way so naturally, as by ascribing it "to their wanting proper opportunities of employing the "money they might save, in some safe and easy method "that would procure them a suitable advantage from it "in the latter periods of their lives. They know, for "the most part, but little of the *public funds*; and when "it happens that they are acquainted with them, the "smallness of the sum they would be entitled to receive, "as the interest of the money they could afford to lay "out in them, is no encouragement to them to dispose "of it in that way. What inducement, for instance, "can it be to a man who has saved ten pounds out of his "year's wages, to invest it in the purchase of 3 *per cent.* "*Bank annuities*, to consider that it will produce him six "or seven shillings a year? It is but the wages of three "days labour. And if they lend their money to tradesmen of their acquaintance, as they sometimes do, it "happens not unfrequently that their creditor becomes a "bankrupt, and the money they had trusted him with is "lost for ever; which discourages others of them from "saving their money at all, and makes them resolve to "spend it in the enjoyment of present pleasure. But if

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(*a*) The amount of the poor-rate for one year at the end of the reign of king Charles II. was £665,362. See *Davenant's* works, Vol. I. p. 38.——In 1777, it was £1,556,804, according to the returns made in that year to parliament by the overseers of the poor.

plans.—But I must leave these hints to be pursued by others.

The

" they saw an easy method of employing the money they " could spare in such a manner as would procure them a " considerable income in return for it at some future pe- " riod of their lives without any such hazard of losing it by " another man's folly or misfortune, it is probable they " would frequently embrace it: And thus a diminution of " the poor-rate on the estates of the rich, and increase of " present industry and sobriety in the poor, and a more in- " dependent and comfortable support of them in their old " age, would be the happy consequences of such an esta- " blishment. Now this might be effected in the following " method."

First, " Let the church-wardens and overseers of every " parish be impowered, by act of parliament, to grant life- " annuities to such of the inhabitants of the parish as shall " be inclined to purchase them, to commence at the end of " one, two, or three years, or such other future period of " time as the purchaser shall chuse, and to be paid out of " the poor-rates of the parish, so that the lands and other " property in the parish that is chargeable to the poor- " rate, shall be answerable for the payment of these an- " nuities. This circumstance would give these annui- " ties great credit with the poor inhabitants, by setting " before them a solid and ample security for the payment " of them."

Secondly, " Let the annuities, thus granted to the " poor inhabitants, be such as arise from a supposition " that the interest of money is 3 *per cent.* or some higher " rate of interest, if the church-wardens and overseers of " the poor think fit to make use of such higher interest."

Thirdly, " But at the rate of 3 *per cent.* the purchaser " should have a right to an annuity, and the church-war- " dens and overseers of the poor should be compellable to " grant it."

Fourthly, " No annuity depending on one life should " exceed £20 a year."

Fifthly, " No less sum than £5 should be allowed " to be employed in the purchase of an annuity.—

" This

The body of dissenting ministers in London had under consideration some time ago

" This is to avoid intricacy and multiplicity in the ac-
" counts.''

Sixthly, " An exact register of these grants should be
" kept, by the church-wardens and overseers of the poor,
" in proper books for the purpose, in which the grants
" should be copied exactly, and the copy of each grant
" subscribed by the person to whom it is granted. And
" this copy, in the register-book of the parish, should be
" good evidence of the purchaser's right to the annuity, in
" case the original deed of grant to the purchaser, which
" was delivered to him at the time of the purchase, should
" be afterwards lost."

Seventhly, " The money thus paid to the church-
" wardens and overseers of the poor for the purpose of
" life-annuities, should be employed in the purchase of
" 3 *per cent.* Bank-annuities in the joint names of all the
" church-wardens and overseers, and by them transferred
" at the expiration of their offices to their successors, and
" so on to the next successors for ever, so as to be
" always the legal property of the church-wardens and
" overseers of the poor for the time being, in trust for
" the persons who should be entitled to the several life-
" annuities, granted in the manner abovementioned; and
" the interest of this money should be received every
" half year, and invested in the purchase of more prin-
" cipal continually, so as to make a perpetual fund for
" the payment of the annuities, &c. &c. Deficiencies,
" if any should ever happen, to be made good by the poor-
" rates, &c. &c."

The very able and public spirited writer of the pamphlet from which I have taken this quotation, now *Cursitor Baron* of the *Exchequer*, took great pains to carry into execution the design he has explained in it. With this view, a bill, with suitable Tables annexed, was brought into the House of Commons and supported by the excellent Sir *George Saville*, the late Mr. *Dowdeswell*, and many others of the most respectable members. It passed that House without much opposition, but was rejected in the House of Lords.

a plan of this kind; and a set of Tables were composed for them. The design was dropped; but as it is possible it may be taken up again, and the Tables may be of use, I have thought fit to preserve them by inserting them among the additional Tables and Observations in the next volume.

Additional Account of the Societies for the Benefit of old Age.

SINCE the publication, in three former editions of the Observations in the preceding Section, almost all the Societies mentioned in it, convinced of the insufficiency and pernicious tendency of their plans, have dissolved themselves, and distributed among their subscribers the money they had paid, with such interest or profit as remained after deducting the expences of management; and there are now left within my knowledge only two of these societies which require any particular notice. I mean, the *Amicable* and the *Laudable* Societies *for the Benefit of Age.* The first of these Societies mentioned in p. 142, finding upon examination that, instead of an annuity of £26, they could not in reality afford to pay a higher annuity than £8, determined, with great fairness, to leave it to the option of all their members, either to continue their contributions with a view to this reduced annuity, or to take back all they had paid and withdraw. Near two hundred members having chosen the former, the Society now consists of *them* only, and therefore can scarcely be in any danger.——The other Society, mentioned p. 141, has also sunk the annuity it promises from £44 to £24; but it is certain, that it cannot permanently pay to all its members a greater annuity than £15. I should lose too much time were I to give an account of the calculations which prove this. He that would see it demonstrated with all possible clearness, should consult a Tract published in 1777, by Mr. *Dale*, entitled, *A* SUPPLEMENT *to Calculations deduced from first Principles*, &c. No person who understands common arithmetic can avoid being convinced by the evidence offered in this Tract, nor can any *honest* man avoid being shocked by the narrative it contains of the obstinacy with which the majority in this Society have persevered in error, contrary to the efforts of the more respectable part of the Society; and in defiance of reason, justice, and humanity. I cannot, in short, speak more properly on this subject than in the words of Mr. *Morgan* in his Treatise *on the Doctrine of Life Annuities and Assurances*, p. 47. "There is ONE Society "for the benefit of old age still left, on "which, none of the calls of justice and "humanity have been able to make any pro-"per impression. I mean the *Laudable So-*

"*ciety* of Annuitants, whose office is held "at the bottom of *Bartholomew-Lane*. In "opposition to the *plainest* evidence, this "Society goes on to offer *double* the annuity "it can afford to pay; and the late trans-"actions in it (as related by Mr. *Dale* in "his *Supplement*) exhibit an instance of "such an obstinate and wilful perseve-"rance in imposition as hath seldom been "equalled.——I am sorry to add," says Mr. *Morgan*, "that this censure is applica-"ble to another Society called also LAUDA-"BLE, but in reality PERNICIOUS, as many "suffering widows will some time or other "experience."

It is here said, that this Society promises *double* the annuity it can pay; that is, £24, when in reality it can pay only £12; whereas I have said that it may pay £16. In order to explain this difference, it is necessary to observe that Mr. *Dale* has shewn that £15 nearly is the annuity which the Society can afford to pay according to the chances of living in Dr. *Halley's* or the *Breslaw* Table; but that the chances of living in the Society had for eight years before 1776[n], (when the number of the Society was above 1300, and its *stock* near £96,000) been found by particular enquiry to be not much less than *double* to those at the same ages in the Table; from whence it follows, that the annuity payable by the Society being more valuable, it ought not to be so high as £15, nor probably more than £12.—The fact now mentioned is important, but not singular; for it has been found to take place in other similar situations[o]; and the reason is, that Dr. *Halley's* and most other Tables of Observations give the chances of living as they exist in towns among men of all sorts taken in the gross; whereas such Societies as those for the benefit of old age, and in general all purchasers of life-annuities for *themselves*, must consist of a *selection* of the best lives.

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[n] I know not what the rate of mortality in the Society has been since this year.

POSTSCRIPT.

Since the preceding Observations were written, I have been informed that this Society has reduced its annuity from £24 to £20. This reduction, together with the high interest at which money may be now improved (particularly in the short annuity), will prolong considerably the duration of the Society; but unless it is favoured by uncommon events, cannot make it permanent[p].

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[o] See additional Observations on Civil Liberty, page 135.

[p] This prediction has been abundantly verified; for, after having gradually reduced their annuities to £8, it appeared

SECT. VI.

Of the Amicable Society for a perpetual Assurance Office: And the Society for Equitable Assurances on Lives and Survivorships.

THE 10th Problem has been given, with a particular view to the corporation of the *Amicable Society*, for a perpetual Assurance-Office on single lives, kept in *Serjeant's-Inn*. This Society was established in 1706, and is

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appeared from an investigation of their affairs which had been made by myself in 1786, that the Society could not then afford to pay their annuitants more than six guineas a year. The number of their members at that time had sunk from 1300 to 204, while their claimants had increased to 564, and their capital which had once consisted of £96,000 stock in the *three per cents*, had been diminished more than one half.—It has, however, been found from subsequent experience that, low as those annuities were then reduced, the circumstances of the Society required that they should be reduced still lower.—In the year 1796, in consequence of a fresh application from this Society, I was engaged a second time in the examination of their affairs.—At that period the annuitants were 471, and the members who had not become claimants amounted only to 24.—The residue of their capital had been changed from *three per cents* into £1660 *per annum* in the Short, and £360 *per annum* in the Long Annuities, which, from the extreme depression of public credit were then hardly worth £17,000.—This depression, more than any other cause, had produced such a deficiency in the accounts of the Society as to render it absolutely necessary to reduce the annuities from 6 guineas to £5.—But they have since been enabled by the improved state of the public funds, and other favourable circumstances, to restore those annuities again to 6 guineas; and it is probable during the remainder of their existence that they will escape the mortification of submitting to another reduction. M.

the only one I am acquainted with, which has stood any considerable trial from time and experience. It is limited by its charter to the disposal of shares or numbers (not to exceed 2000) held by *single* lives, and entitling to *claims* when the lives drop. For each of these shares every purchaser pays at entrance £7...10*s.* besides £1...11*s.* as the first *quarterly payment* of £6...4*s. per ann.* to be continued during life. An annual dividend of £1...4*s.* for each share is allowed to every purchaser out of the profits of the corporation, which reduces the *annual* payment for each share to £5. The net annual income arising from all the annual payments (making £10,000 when the Society is full, and all the shares are disposed of), is equally divided among the nominees of such members as die within the year; which dividend, therefore, is more or less at the end of every year as a smaller or greater number of the members happen to die in that year. In 1757, the Society engaged that this dividend though it might be more, should not be less on each share than £125, and in 1770[q], that it should not be *less* than £150.——No one person is allowed to purchase more than

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[q] In 1757, the Society had accumulated by its savings £25,300 three *per cent.* stock, which in 1770, had been increased to £33,800; and a part of this stock was in these years appropriated to the payment of claims, whenever the number of them in any year should happen to be so great as to render the annual contributions insufficient to make them up to the guaranteed sums.

three shares; nor are any admitted to be purchasers whose ages exceed 45, or fall short of 12; and all between these ages are admitted on the same terms.

This Society has, I doubt not, been very useful to the public; and its plan is such, that it cannot fail to *continue* to be so. It might, however, certainly, have been much more useful, had it gone from the first on a different plan. It is obvious, that regulating the dividends among the *nominees* by the number of members who die every year, is not *equitable;* because it makes the benefit which a member is to receive to depend, not on the value of his contribution, but on a *contingency;* that is, the number of members that shall happen to die the same year with him. This regulation must also have been disadvantageous to the Society; as will appear from the following account of the natural progress of the affairs of such a Society, when established on a right plan.

Suppose a *thousand* persons whose common age is 36, to form themselves into a Society for the purpose of *assuring* a particular sum at their deaths, to such persons as they shall name, in consideration of a particular annual contribution to be continued during their lives. Suppose the annual contribution to be £5, and the first payment[1] to be made immediately. Suppose, likewise the original number of the Society to be constantly kept up by the admission of new members, at 36 years of age, in the room of such as die.—In Question X. p. 32, it appears, that an annual payment, beginning immediately, of £5 during a life now at the age of 36, should entitle, at the failure of such a life, to £172, reckoning interest at 4 *per cent.* and taking Mr. *De Moivre's* valuation of lives.—A *thousand* persons, all 36 years of age, will die off at the rate of 20 every year. The disbursements, therefore, of such a Society will be, the first year, 20 times £172, or £3,440, and its income will be £5000. It will, therefore, at the end of the year, have a surplus of £1,560 to put to interest.—In consequence of the yearly accessions to supply vacancies, the number dying annually will be always increasing after the first year. In 50 years[s] it will get to a *maximum*; and then the affairs of the Society will become *stationary*, and the number dying annually will be 40, and its annual expence will be £6,880 exceeding the annual contribution, £1,880. But, in the mean time, by improving its surplus monies, it will have raised a capital equal to this excess, and, consequently, its affairs will be fixed on a firm basis for all subsequent times.

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[1] Such payments, it has been shewn, Question VIII. p. 29, are better than any *half*-yearly or *quarterly* payments, and at the same time they save some trouble.

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[s] This period will (by Question III.) be longer if the Society is any time in filling, and admits members at younger ages than 36. It will, for instance, be 84 years, if the Society is ten years in filling, and admits at all ages between 12 and 45.

Suppose now that such a Society, at its establishment, should resolve to divide its whole yearly income among the *nominees* of deceased members. The effect of this would be, that no capital could be raised; that the dividends payable to *nominees* would diminish continually, till, at the time that the greatest number of members came to die *annually*, (or at the end of 50 years), they would be reduced to *half*; and that all claimants, after this period, would receive too little, because the first claimants had received too much[t].

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[t] The reverse of this will take place, if such a Society *begins* with admitting at all ages, and afterwards changes its plan, and *limits* the age of admission. In this case, the number of *yearly deaths* will be *greatest* at first, and the *dividends smallest*. In consequence of altering its plan, the *yearly deaths* will lessen gradually, and the *dividends* rise; but in time *both* would return again to their original state.

The following facts incline me to suspect, that this remark may be applicable to the *Amicable Corporation*.

First. In their *original charter*, as it is given in their printed abstracts, there is no limitation of age mentioned; but 31 years afterwards, I find a bye-law made against admitting any person who should be above the age of 45, or under 12.—Secondly. In their printed advertisements in 1770, it is said, that in 59 years they had paid, among 8,643 claimants, £378,184, from whence it follows, that though the average of their dividends, for 17 years before 1773, has been £154, the average for 59 years, has been only £104.

At the time of the institution of the *Amicable Corporation*, the interest of money was at 6 *per cent.* and, as they admit none whose ages are not under 45, the mean age of admission cannot be much greater than 36. It appears, therefore, that had they avoided the error now mentioned, and gone from the first on the plan I have described, they might have all along paid to each *nominee* £172, besides raising a capital much greater in proportion to the number of members, than that I have specified, from the premiums at admission, forfeitures, and other advantages which they have enjoyed[u]. Indeed, I cannot doubt but that on this plan, and with these advantages, they might have found themselves always able to pay at least £200 to each *nominee*[x].

I have already mentioned one instance in which the plan of this Society is not equitable.

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[u] A surplus from a *thousand* members of only *five shillings per annum*, duly improved at 4 *per cent.* would, in 41 years, produce a capital of £25000.

[x] It should be remembered, that all this was said in the former editions, on the supposition that proper care has been taken to keep out unhealthy persons; that the ages of admission have never exceeded 45; and that the probabilities of life among the members of this Society, are the same with those in the 5th, 8th, and 17th Tables, in the next volume. But I have lately found the truth to be, agreeably to the suspicion expressed in the last note but one, that for many years after the first institution of the Society, members were admitted at all ages.

able. Another instance of this is, their requiring the same payments from all persons under 45, without regarding the differences of their ages; whereas, the annual payments of a person admitted at 45, ought to be double the annual payment of a person admitted at 12.

Further. The plan of this Society is so narrow, as to confine its usefulness too much. It can be of no service to any person whose age exceeds 45. It is, likewise, far from being properly adapted to the circumstances of persons, who want to make assurances on their lives, for only short terms of years.—Thus; the true value of the assurance of £150 for 10 years, on the life of a person whose age is 30, is by Quest. XIV. (interest being at 3 *per cent.*) £2...13*s.* in annual payments (for 10 years), to begin at the end of the first year; and subject to failure when the life fails. But such an assurance could not be made, in this Society, without an annual payment of £5.—Neither is the plan of this Society at all adapted to the circumstances of persons, who want to make assurances on particular survivorships.—For example: A person possessed of an estate, or salary, which must be lost with his life, has a person dependent upon him, for whom he desires to secure a sum of money, payable at his death. But he desires this only as a provision against the danger of his dying *first*, and leaving a wife, or a parent, without support. In these circumstances, he enters himself into this Society; and by an annual payment of £5 entitles his *nominee* to £150. In a few years, perhaps, his *nominee* happens to die; and, having then lost the benefit he had in view, he determines to forfeit his former payments, and to withdraw from the Society. In this way probably, this Society must have gained some advantages. But the right method would have been to have taken from such a person the true value of the sum assured, "on the "supposition of non-payment, provided he "should survive." In this way he would have chosen to contract with the Society; and had he done this, he would have paid for the *assurance*, (supposing interest at 3 *per cent.* his age 30, the age of his *nominee* 30, and the probabilities of life as in the 5th, 8th, and 17th Tables) £3...8*s*.[y] in annual payments to begin immediately, and to be continued during the *joint* continuance of his own life, and the life of his *nominee*.

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[y] The value of £150 payable at the death of a person, aged 30, *provided* he survives another person of the same age, is, by Question XI. Chap. I. £45.65; and this value divided by 13.43 (the value increased by unity, of two joint lives both 30), gives £3.4, or £3...8*s*.—The value of the same reversion, according to the probabilities of life in *London*, is, £49.19, in *one* payment; and £4.16, in *annual* payments, during the joint lives, the first payment to be made immediately.

Further Account of the Amicable Corporation.

THE affairs of this *Corporation* have lately taken a very favourable turn. The dividends, from the annual contributions of £5 for each number, which for eight years, ended in 1769, had not been £150 on each claim, have, for eight years ended at 1779, been nearly £200[z].——The subsisting shares have increased from 1120, (their number in 1769) to 1990 (their full complement nearly) in 1780; and in the same time the *stock* of the Society has been increased from £33,300 to £51,300 in the 3 *per cent.* annuities; in consequence of which it finds itself now possest, after discharging all expences, of a clear surplus of about £1,350 *per ann.*——In these circumstances a proposal has been made to the Society to discontinue the increase of its stock, in order to make use of the surplus in increasing the dividends on claims. This being an inviting proposal, it is not surprising that the Society in general has shewn itself disposed to accede to it.——The imprudence however of such a step will be evident from the following observations.——It should be considered, that the reason of the late increase of the annual dividends, has been the late increase of the Society by the influx of young members. So great has been this increase, that the Society has been nearly *doubled* in twelve years, and *tripled* in the last 30 years.——It could not, therefore, but happen that the number of deaths should become much less in proportion to the number of members, than they were *before* the increase. The Society being now full, and admitting of no farther increase, the collective age of the members, and, together with it, the annual deaths, will be for some time increasing, till that part of the Society which consists of the late additions[a] come to die off *twice* as fast as they did at first.

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[z] This dividend for 1780, was £192...6*s*...1 ¾*d.*; but made up to £193 in consequence of a discretionary power given the directors to take what sums they think reasonable from the savings to increase the dividends.

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[a] An addition of near 400 to the subsisting numbers or shares of the Society was made from 1749 to 1768; and of above 900 more, from 1769 to 1779.—Even that part of the increase of the Society which consists of members to whom the first of these additions has been owing, is at present far from dying off so fast as it will. In truth, 50 or 60 years at least must elapse before this can happen. See Question III. Chap. I. I am told it has been asserted by some belonging to the Society, that a person about 40 or 45 is not less likely to die than a person 20 or 30 years older, from whence, I suppose it is inferred, that the deaths now in the Society may be as numerous as they will ever be. It would be doing an injury to the Society to suppose that it can be influenced by

During this interval the dividends will be growing less and less, till at last they will fall below the dividend which the Society has guaranteed (or £150), and produce a necessity of entering into the capital in order to make it up. And the reduction of the capital once begun will proceed faster and faster till it is all spent; and when spent, the Society will be thrown back into the state it was in before its increase, when frequently it could not afford a dividend of £100 on each number.—Such will be the certain effect of adopting the measure I have mentioned. It is, therefore, imprudent in the highest degree; and I will add, that the *injustice* of it is equal to its imprudence; for it is benefiting the *older* part of the present members at the expence of the *younger* members, and all *present* members at the expence of *future* ones. For a few years the dividends will probably, with the help of the addition of the annual profits, exceed £200, which will be paid chiefly among the claimants derived from the older members.

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by such assertions. According to all observations on human mortality, a body of men at 65 or 70 will die off twice as fast as a body of men at 40; and a body of men at 75 or 80 will die off *four* or *five* times as fast as a body of men at 50. The human frame, after the age of 12 or 15, is continually wearing out and becoming less capable of combating the causes of mortality; but more or less slowly according to the degrees of firmness with which it was built, and the favourableness or unfavourableness of the situations into which it happens to fall.

bers. But after a course of years they will sink to little more than half; and at that period the Society will consist of the younger part of the present members, and such *new* members as shall be hereafter admitted to fill up vacancies, who will therefore be great *losers*, because their predecessors, by neglecting to improve the estate, made themselves too great *gainers*.

In order to set this in a clearer light, I would desire it may be considered that according to the mean probabilities of the duration of life, a body of people at the age of 36, will, one with another, live 25 years. It must, therefore, be expected that a *twenty-fifth* part will die annually of a Society which has subsisted any considerable time, and the members of which are admitted at this mean age. Supposing, therefore, 36 the mean age of admission in the *Amicable Corporation*, the time must come when (if kept[b] up to its full complement) a 25th part of the members will die annually, or when 80 numbers (the 25th of 2000) will drop, and produce 80 claims annually.—The whole income of the Society (consisting of £10,000 *per ann.* from the charter contributions, and £1,350 *per annum*, profits), when divided equally among 80 claimants, will give £142 for each claim. The dividend guaranteed in

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[b] Should the Society decline, more than a 25th part will die annually, and the period when this will happen will arrive sooner.

1770 being £150 there will be a deficiency of £8 in each claim, or of £640 in the total of claims; which sum procured by selling stock in one year will leave a necessity of selling more the next year, and still more the following, and so on through every successive year, till the whole stock falls rapidly to nothing.——It is proper to observe here, that this deficiency of £640 *per ann.* is a *maximum* to which the deficiencies of many preceding years had increased gradually; and that, therefore, the whole stock may be consumed even before the period arrives, when the greatest deficiencies will happen.

All this reasoning supposes that no more than a 25th part of the members of this Corporation will hereafter come to die annually; or that they are admitted at the mean age of 36, and live after admission, one with another, 25 years.—But these are probably too favourable suppositions. Perhaps, the mean age at which members are admitted (and particularly the purchasers of two or three numbers) may be above 40; and perhaps also, on account of the difficulty there must be in excluding from *such institutions* all bad lives, the duration of the lives of the members may be somewhat less than is common among persons at the same ages. Should this be true, their duration of life after admission will not exceed 22 years. A 22d part will hereafter die annually. The claims from 2000 numbers will be 91; and the dividend on each claim, with the addition of the savings, will be only £124...12*s.* which will produce a deficiency of £2,311 *per ann.*

If I may judge from what has hitherto happened in the Society, even these last suppositions favour it too much; for, I find, that before the increase [c] which begun in 1750, a *nineteenth* part of the existing numbers dropped annually, which made the dividends then fall frequently below £100. This, however, must have been owing to the admission, for some years after the establishment of the Society, of members at too advanced ages, and the neglect of proper care to exclude bad lives. Much uncertainty in this instance would be removed, and the best guide obtained in conducting the affairs of the Society, by taking an exact account of the mean age at which, for the last ten or twelve years, all members (distinguishing particularly such as have two or three numbers dependent on their lives) have been admitted; and also of the number of years which all admitted, till within the last 20 years, have lived after admission.

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[c] This increase seems to have been owing to the limitation of age in 1737. In a few years after this, the greatest part of the members who had been admitted at advanced ages having died off, and the Society consisting chiefly of younger members admitted in their room; the dividends rose, which occasioned a quick *increase* of young members, raised the dividends still higher, till, in 1757, the Society thought itself capable of guaranteeing a dividend of £125; and, in 1770, a dividend of £150. These augmentations contributed yet more to increase the Society, and consequently the dividends. It became, therefore, soon full, and now finds itself in the prosperous state described at the beginning of these Observations; a state, which the Society may render *really* and *permanently* prosperous, if they are properly attentive to its causes, and will avail themselves of the opportunity it gives them to rectify the faults in their original plan by increasing their guaranteed dividend in the manner I shall presently propose, instead of discontinuing the increase of their capital.

Upon the whole. Till new light is given by such an enquiry as this, I must think that, however prosperous the affairs of the Society seem at present, it cannot prudently act on any other expectation than that a period will come when a 22d or 23d[d] part of its members will die annually, and when, therefore, it will want at least an *additional* income of £2000 *per ann.* to enable it to make good its engagements. A greater additional income might be acquired by continuing to lay up all its savings after dividing the charter contributions—But were it to lay up the *whole of its income* above what may be necessary to divide £170 on every claim, it might advance its *guaranteed* dividend from £150 to *this sum.* And this, in my opinion, is the most rational and equitable measure it can adopt.

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[d] If Mr. *Brand*, the late clerk of this Society, is right, the members of it are uncommonly short-lived; for, according to an account which he has given of the duration of the lives of 3,826 persons who have been admitted into it, and on whose deaths claims have been paid; they do not, one with another, live after admission 16 years. See *Smart's* Tables republished by Mr. *Brand*, p. 189. See likewise his Treatise on Assurances and Annuities, p. 68.

This, were it true, would be very threatening to the Society. But Mr. *Brand*, in other parts of the *little* that is his own in these Treatises, has erred so palpably, and shewn himself so wonderfully ignorant, that, however possessed in this instance of the means of information, he deserves no credit.—The truth is, in the present case, that he has included in his account such members as have been admitted and have died *lately*; and this must necessarily lead to a wrong conclusion concerning the duration of the lives of the members taken at large.

I know, indeed, that there is a clause in the charter which limits the increase of its estate to £2000 *per ann.* But this clause shews that the charter was framed with too little foresight; and it must be repealed, or the consequences will be that *danger* to the Society and *injustice* to its future members which I have represented.

The preceding Observations are offered very respectfully to the consideration of this Society. Should any of the members or directors think them worth their attention, I hope they will reflect, that having no interest to serve, I can mean nothing by them but the preservation of the credit and usefulness of the Society[c].

SECT.

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[c] It does not appear that any attention has been paid to the admonitions and advice which were given in this Section.—When the Society stipulated in 1770 that the claims should never be less on each share than £150, the number of subscribers, as might naturally be expected, increased very rapidly: By the accession of so many new members a sudden addition was made to their annual income,

SECT. VII.

Of the Society for Equitable Assurances on Lives and Survivorships.

THE Society which is to be the subject of this Section, has justly styled itself, "*A Society* "*for Equitable Assurances on Lives and Sur*"*vivorships.*" The business of it is carried on at its office, in *Chatham-Place*, near *Blackfriars Bridge*. It was founded in 1762; in

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income, and, in consequence, the dividends on the claims were increased so as to raise them during the first 10 or 15 years from £150 to £200, and in some years even to £300. But this effect necessarily ceased with the cause which produced it, and the dividends since the year 1787 have sometimes fallen so low as barely to exceed the stipulated sum of £150.—Nay, in the year 1802 they fell short of this sum, and the Society had recourse to their capital, not merely to supply the deficiency, but even to add £5 to each share over and above the sum which they had engaged to pay. In some years they have added £15; nor do they seem in these additions to be guided by any one principle of reason or computation. It is at best a very mistaken liberality, and if it were always to be repeated under similar circumstances, though it might perhaps be intended to promote the credit, it would inevitably terminate in the ruin of the Society.

By increasing the number of their shares about 20 years ago from 2000 to 4000, they produced in like manner as in the year 1770, a temporary increase of the dividends: but the effect in this instance was more transient, owing, I believe, principally to the new members having for the most part consisted of persons between 40 and 50 years of age.—In the admission of these members they raised the contributions upon every number above 2000 from £5 to £6. 4*s*. *per annum*: thus, making the amount of the annual payments to depend, not on the age of the subscriber, but on his priority in the Society.—It is much to be lamented that they should not have chosen to in consequence of lectures recommending such an institution, which had been read by

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to adopt the improvements proposed in this work, rather than persevere in a course so injurious to their real interest, and often disgraced by regulations so absurd and inequitable. About 15 years ago they manifested a disposition to reform their plan, and in consequence of the recommendation of the Directors they limited the age of admission to 40 instead of 45—they also placed all the members on the same footing in regard to their annual payments--restrained for a time the amount of the dividends to £150—and, in order to encourage young persons to subscribe, they gave those who entered under the age of 35, the privilege of having six, instead of the usual number of three shares, on their respective lives:—They also agreed that £34,300 of their stock, (amounting in the whole to £64,300 in the *three per cents*, together with some property in houses) should be preserved as unalienable without the consent of two successive general courts, so that only £30,000 of their stock should be appropriated in aid of the annual contributions to make up future deficiencies in the dividends.—Had these regulations been proposed at an earlier period, it is possible that they might have been effectual; but I have every reason to believe from an investigation of the affairs of this Society at that time, that they were much too feeble and inefficient.—The reform of a plan originally very defective, and rendered still more so by subsequent laws and alterations, is indeed a work of considerable difficulty, and if the members satisfy themselves with such measures as they then adopted, it is certain that they are far from having accomplished such a reform as will secure the Society from the consequences of their former errors.

The Society has lately obtained an extension of their Charter, by which they are empowered to assure persons of all ages at nearly the same premiums with those of the Equitable Society. This has necessarily, by the accession of new members, increased their capital, and they have, in consequence, determined that the dividends on the assurances, according to their original plan, shall be raised to £200!! M.

Mr. *Simpson*, a name that can never be forgotten while there is any mathematical or philosophical knowledge left in the world. Mr. *Dodson* also, the author of the *Mathematical Repository*, was active in recommending the plan of this Society, and composed Tables for its use.—It assures any sums or reversionary annuities on any lives, for any number of years as well as for the whole continuance of the lives, at rates settled by particular calculation; and in any manner that may be best adapted to the views of the persons assured. That is, either by making the assured sums payable *certainly* at the failure of any given lives, or on condition of survivorship: and also either by taking the price of the assurance in *one present payment*, or in *annual payments* during any single or joint lives, or any terms less than the whole continuance of the lives.—In short; the plan of this Society is so extensive and so important, that I cannot satisfy my own mind, without offering to the gentlemen concerned in the direction of it, the following observations, hoping they will not think them impertinent.

First. They should consider what distress would arise from the failure of such a scheme in any future time; and what dangers there are, which ought to be carefully guarded against in order to secure success. I have already more than once observed, that those persons will be most for flying to these establishments, who have feeble constitutions, or are subject to distempers which they know render their lives particularly precarious; and it is to be feared, that no caution will be sufficient to prevent all danger from hence.

Again. In matters of chance, it is impossible to say, that an unfavourable run of events will not come, which may hurt the best contrived scheme. The calculations only determine probabilities; and, agreeably to these, it may be depended on, that events will happen on the whole. But at particular periods, and in particular instances, great deviations will often happen; and these deviations, at the commencement of a scheme, must prove either very favourable, or very unfavourable.

But further. The calculations suppose, that all the monies received are put out immediately to accumulate at compound interest. They make no allowance for losses, or for any of the expences attending management. On these accounts, the payments to a Society of this kind, ought to be more than the calculations will warrant, and the interest of money ought to be reckoned low. Mr. *Dodson,* I find, has paid due attention to all this, by reckoning interest, in his calculations for this Society, at 3 *per cent.* and taking the lowest of all the known probabilities of life, or those deduced from the

London bills of mortality[f]. There is, besides, a liberty provided of making a call on all the members, in case of any particular emergency. It is, therefore, highly probable, that this Society must be secure. The last expedient, however, would be a very disagreeable one, should there be ever any occasion for having recourse to it; and, in order to guard still more effectually against danger, it would not, I think, be amiss to charge a profit of 6 *per cent.* on all the payments. Should the consequence of this prove, that in some future period the Society shall find itself possessed of too large a capital, the harm will be trifling, and future members will reap the advantage. But this leads me to repeat an observation of particular consequence.

As this Society is guided, in every instance, by strict calculation, it is not to be expected that it can meet with any difficulties for many years; because not till the end of many years after it has acquired its *maximum* of members, will the *maximum* of yearly claimants and annuitants come upon it. Should it, therefore, through inattention to this remark, and the encouragement arising from the possession of a large surplus, be led to check or stop the increase of its stock too soon, the consequences might prove pernicious.

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[f] It ought, however, to be remembered here, that in selling life-annuities to commence either immediately, or after given terms; and also in some other cases, the values come out *less* in consequence of *lower* probabilities of life. Would it in *such* instances be taking an unfair advantage, to estimate the values by Tables which give the *highest* rather than the *lowest* values? Thus; were the Society to sell £20 *per annum* for life, to a person now 30, after 50, the value, according to Dr. *Halley's* Table, would, reckoning interest at 3 *per cent.* be £90 in a single payment; but according to the *London* Table, the value would be only £70.

But in reality the value, even by Dr. *Halley's* Table, is less than the Society, in such a case, ought to take, for the reason mentioned in page 157.

Again. I would observe, that it is of great importance to the safety of such a Society, that its affairs should be under the inspection of able mathematicians. Melancholy experience shews, that none but mathematicians are qualified for forming and conducting schemes of this kind.—In short; dangerous mistakes may sometimes be committed, if the affairs of such a Society are not managed frugally, carefully, and prudently. One instance of this I cannot avoid mentioning.

A person, who desires to assure a particular sum to be paid at the failure of his life, on condition of the survivorship of another life, may chuse to pay the value in annual contributions during the continuance of his own single life rather than during the continuance of the joint lives, because the annual contributions, in this case, ought to be much less. But a Society that would practise such a method of *assurance* would hurt itself; for as soon as the life, on whose survivorship the assurance depends, is extinct, the person assured, if then living, would have no longer any benefit in view; and, therefore, would make his payments with reluctance, and in time, perhaps, entirely withdraw them; the consequence of which would be, that the Society would suffer a loss by being deprived of the just value of the expectation it had granted. The plan of a Society ought always to be such, as that the losses arising from discontinuance of payment, should fall on the purchaser, and never on the Society.

I must not forget to add, that it is necessary that such a Society should be furnished with as complete a set of Tables as possible. This will render the business of the Society much more easy, and also much more capable of being conducted by persons unskilled in mathematics. It will also contribute much to its *safety*. For in all cases to which Tables can be extended, there would be no occasion for employing any calculators; and, consequently, a danger would be prevented to which, though it is not *now*, it may *hereafter* be exposed; I mean, the danger of happening to trust unskilful or careless calculators.—It is indeed furnished with Tables, by which a great part of its business is transacted; but there are some important Tables which it wants, and with which it should be supplied; and these when composed, together with all its other Tables, should be subject to the revisal and examination of the best judges, and afterwards published; with a minute account of the principles assumed and the method taken in composing them. Such a publication would be a valuable addition to this part of science; and it would also be the means of increasing and establishing the credit of the Society.

In Questions 4th, 6th, 10th, 11th, 14th, 15th, and 16th, I have, with a particular view to this Society, given rules by which may be formed every Table it can want, for shewing the values of assurances on the *whole duration,* or any *terms,* of any *one* or *two* lives; and nothing but care and attention can be necessary to enable any good arithmetician to calculate from them.

Further Account of the Equitable *Society, with an Account of an Institution for the Sale of Life-Annuities at* Hamburgh.

I HAVE just referred to the questions in the first Chapter of this Treatise, for the rules by which the values of assurances on any *one* life, or any *two* lives, may be computed. Since the last publication of this Treatise, investigations of this kind have been carried much farther, and this subject, as far as it respects assurances on any number of lives not exceeding *three,* has been nearly exhausted by Mr. *Morgan,* the actuary of this Society, in the Treatise referred to at the end of the First Chapter.—In this work, Mr. *Morgan* has given a distinct account of the state of this Society as he had made it out to January 1777; and to that account, and to the Observations addressed to the Society in the Introduction to Mr *Morgan's* Treatise, I must refer for the fullest information that can be given of the plan and progressive increase of the Society, and of the methods employed to keep in constant view the state of its accounts. I shall here only add, that its increase has been going on ever since with rapidity; that, in the last five years (or since 1776), its *annual income* has been nearly *doubled*, and its *capital* (consisting now in part of land securities) more than *tripled*; that the rate of mortality among the persons assured has continued much below that in the Tables by which it has hitherto made its calculations;[g] and that, upon

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[g] Until the time nearly in which this account was written the Society had computed all their premiums from the *London* Table of Observations, and it appeared from the experience of twelve years, (or from 1768 to 1780) that the decrements of life among their members had been

	To those in *London.*	To those in *Breslaw.*
from the age of 10 to 20 in the ratio of	1 to $1\frac{1}{4}$	1 to 1
20 to 30	3 to 7	2 to 3
30 to 40	4 to 9	2 to 3
40 to 50	3 to 5	8 to 9
50 to 60	2 to 3	6 to 7
60 to 70	9 to 10	9 to 8

upon the whole, it appears at present to possess such a *surplus* of income and stock as places it (if no mismanagement takes place) above danger, except from events the most extraordinary.

In these circumstances, the Society, not willing to raise an exorbitant capital, or to take unreasonable profits, came to a resolution at the beginning of the last year, or 1781, to make such abatements in its demands as its present circumstances render safe, and to settle such new arrangements in its business as may contribute to make it as great a benefit as possible to the public.

In the preceding Section, but more particularly in the Introduction to Mr. *Morgan's* Treatise, I have expressed my wishes that the Society would order new Tables to be calculated from Observations more adapted

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In consequence they determined to compute the premiums in future from a table which should give the probabilities of life *higher* than that which they had hitherto used; and for this purpose they adopted one which had been just formed by Dr. *Price* from very accurate observations made in the town of *Northampton*. Compared with the decrements of life in that table from the year 1768 to the year 1810, the decrements of life in the Society appear to have been from the age of

10 to 20 in the ratio of 1 to 2
20 to 30 . . . . 1 to 2
30 to 40 . . . . 3 to 5
40 to 50 . . . . 3 to 5
50 to 60 . . . . 5 to 7
60 to 80 . . . . 4 to 5 or at all ages together in the ratio of *two* to *three*. M.

to the general state of mortality among mankind than those given by the *London* bills of mortality.——I can now inform the public, that such an order has been given and lately carried into execution.——These new Tables are,

First. A Table exhibiting the values of single lives for their whole duration.

Secondly. A Table of the values of single lives for any terms of years not exceeding ten.

Thirdly. A Table of the values in single and annual payments, of assurances on single lives for terms, and for their whole duration

Fourthly. A Table of the values of two joint lives for all ages.

Fifthly. A Table of the values, in single and annual payments; of assurances of gross sums, and life annuities payable on the survivorship of one life beyond another.——The most material parts of these Tables will be found among the other Tables in the next volume. They have been calculated by Mr. *Morgan* with incredible care and industry; and are correct and complete to a degree never before attempted in any Tables of this kind. They are to form the basis of the future business of the Society, and must conduce much to its growing credit and usefulness.

The second, third, and fifth Tables, have been calculated by the rules in Chap. 1st of this Treatise, Question 6th, 10th, 11th, and 14th. And the other Tables, or the 1st and 4th just mentioned, in the method described by Mr. *Morgan* in the 2d Section of the second Chapter of his Treatise on the Doctrine of Annuities and Assurances; a method which, at the same time that it lessens the labour of these calculations, prevents the possibility of falling into any mistakes[h].

They are all founded on a Table of the probabilities of the duration of human life at *Northampton*, which will be inserted among the other Tables in the second volume of this work. This Table made a part of all the former editions of this work; but it is, in the present edition much improved, and gives, I believe, more correctly than any other, the mean probabilities of the duration of human life; and, therefore, as I shall observe again hereafter, seems to be more proper than any other for general use.

I had, in the Introduction to Mr. *Morgan's* Treatise, recommended to the Society the observations on human mortality at *Chester*; and I had procured a copy of them from Dr. HAYGARTH, the ingenious founder of them. But the directors of the Society have judged very rightly, that they carry the probabilities of life too high for their business.—These Observations, however, are not on this account less important. I have been enabled by them to make the *Northampton* Table of Observations more complete; and Tables of the decrements and expectations of lives, deduced from them, for both sexes, will be given in the next volume.

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[h] *Specimens* of the shortest and easiest method of making these calculations by *Logarithms*, and grounded on Mr. *Morgan's* method here mentioned, will be given in a Postscript to the 4th Chapter.

The interest of money in calculating the new Tables of the Society has been reckoned, as it was in the old Tables, at 3 *per cent.* This gives the Society (more especially at present when money may be improved at near double this interest) a very great advantage. It likewise possesses the two following advantages.

First. The interest of a large and fast increasing capital, the greatest part of which is a SURPLUS over and above all that is necessary to enable it to make good its engagements.

Secondly. The profits arising from higher probabilities of living among the members of the Society than are exhibited, even in the new Table of Observations, by which its demands are for the future to be governed. This Table differs but little from the *Breslaw* or Dr. *Halley's* Table of Observations, which, as may be seen in the Note, p. 182, gives probabilities of living near a *third* lower than those which have hitherto taken place in the Society.

I believe the Society might now safely trust itself to the security arising from these advantages, and take the payments for assurances in strict conformity to its new Tables, without any charge upon them; and the consequence of this would be, that these payments, which four years ago were reduced a *tenth*, will be further reduced about *two tenths*, or in the whole about 30 *per cent*[i].

But as it has been the custom of the Society (in conformity to the recommendation in p. 178) to make an addition of 6 *per cent.* to all the payments required by the old Tables, it may, I think, be excused, if, for the sake of greater safety and to provide better for the expences of management, it should make an addition of 3 or 4 *per cent.* to the payments required by the new Tables.

There still remain a few Tables, which perhaps some time or other the Society may think proper to furnish itself with.——I will mention the two following.

First. A Table of the same kind with that mentioned in p. 135, shewing the values of sums payable *at* a given age, and of annuities payable *till* a given age, to a child, should he lose his parent.

Secondly. A Table containing the values

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[i] It should be remembered, that the reduction here mentioned, does not extend to the *single* payments for assurances of *life-annuities* payable on *survivorship*, the *London* Table of Observations giving these almost as low as the *Northampton* Table.

of assurances of annuities for the remainder of life after given terms.

This would inform the public what advantages could be reasonably offered to persons who wish to be purchasers of such annuities; and the avidity with which the deceptions in this way have been encouraged renders such an information particularly proper.——The public, indeed, has been led by these deceptions to entertain such wrong ideas of the terms on which these annuities may be sold, that probably no terms which the Society can afford will appear sufficiently encouraging. There are, however, annuities of this kind which, at the same time that they have the most useful tendency, might possibly invite purchasers.——Suppose, in particular, an annuity of £1 to commence at 56, and to increase at the rate of £1 every year afterwards, (so as to become £15 at 70, £25 at 80, and £35 at 90) was offered for a given sum payable at the age of 30, with proper abatements for every year that the purchase was made before this age.—Would not such a proposal be likely to engage attention? And might it not be extremely useful, by holding forth an incitement to industry in the beginning of life, and providing a way of laying out small savings to the best advantage?——According to a high valuation the sum in this case would be about twenty guineas; and the proper abatement about twenty-five shillings for every year that a purchaser's age, if not less that 20, falls short of 30.

I must not conclude these Observations on annuity schemes without taking particular notice of an excellent example in this way, which is given by a general annuity institution lately established at *Hamburgh;* and to which I have referred at the end of the fourth Section, p. 137.

Having received, through Mr. *Oeder* at *Oldenburgh*, an account of this institution, and finding that the conductors of it wish to extend its advantages beyond the limits of *Hamburgh*; I embrace with pleasure this opportunity of recommending it, and reciting the following particulars in its plan.

Persons of all ages who may desire to increase their incomes by purchasing annuities for their own single lives, or for the longest of any two lives, may in this institution purchase such annuities.——A person at 50 may receive during his life $7\frac{5}{8}$ *per cent.* for any sum; at 60, he may receive $10\frac{1}{16}$ *per cent.*; and 70, 15 *per cent.*——Persons who depend for a subsistence on the permanency of their capacities for service or labour, may with such savings as they may be able to make in their years of vigour, purchase for themselves a competence for old age A person at 40 may with £100 purchase £18 *per ann.* for his life after 55; or, for a payment of 17*s*...6*d.* every half year till he is 60, he may purchase for his life after that age £6...12*s. per ann.*——Young persons whose fortunes do not produce a sufficient income may, by sinking a part of them, procure the means of a future settlement in life. A boy, for instance, aged 10, may with £100 purchase an annuity of £8¼ dependent on his life till he is 25 years of age, when it may be supposed he will be provided with other means of supporting himself.

Persons who have friends or relations dependent upon them may purchase for them, either by a single present payment, or by half-yearly payments, any annuities to commence at the time of their survivorship, should that happen, and to be continued during the remainder of their lives. In the case of widows an abatement is made, if the purchaser chuses that the annuity should be paid only during widowhood.

Parents wishing to provide *portions* for their children, or sums for putting them out to apprenticeships when grown up to a certain age, may purchase (by either half-yearly contributions or single payments), such portions or sums to be paid them at that age, should they live to it.

The plan of this institution includes in it several other particulars; but I will only add, that the money received by the conductors of it is lodged in the chamber of *Hamburgh;* that the prices or contributions are distinctly specified for every age in a set of Tables which have been published at *Hamburgh*; and that these Tables have been calculated at an interest of 3 *per cent.* from some of the best registers of mortality, and (as far as I have examined them) with skill and correctness[k].

CHAP.

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[k] The Equitable Society, since these observations were written in the year 1783, has increased so immensely both in the number and magnitude of its assurances, as to become an institution of the first importance. In the year 1786, after a minute computation of the value of each separate assurance, the addition of 1½ *per cent*, which had been made to the claims in 1782 for every annual payment prior to that year, was increased to 2½ *per cent.* In the years 1791, 1793, and 1795, still farther additions were made, amounting together to 4 *per cent*; so that the claims on all assurances of an earlier date than 1771 were more than *doubled*, and even those of so late a date as 1783 were increased nearly *one half*. But although the Society never hazarded an addition without a thorough conviction of its safety founded on a comparison of the claims with the premiums and on the decrements of life among its members in each year, it wisely determined to proceed no further with these additions till the same investigation of its affairs had taken place, which had been first instituted in 1776, and repeated in 1786.—Accordingly, in the year 1800 this work (which from the increased number of assurances had become a most arduous and toilsome undertaking) was accomplised, and the result proved so highly favourable that another addition of £2 *per cent.* was made to the claims, so that at that time the assurances of 1790 were increased £36 *per cent.*; those of 1780 £105 *per cent.* those of 1770 £190 *per cent.* and those of 1762 £258 *per cent.* making in the whole a sum which in *present value* fell very little short of *half a million!*—In 1809 the same investigation was repeated, which, from the immense increase in the number of its assurances, was attended with infinitely more labour and difficulty. The result, however, like that of all former investigations, proved that the success and opulence of the Society increased in proportion to the magnitude of its concerns, and an addition

CHAPTER III.

Of the Method of calculating the Values of Reversions depending on Survivorships.

ALL Questions relating to the values of lives and reversions, are at present of particular importance in this kingdom. Much business is continually transacted in this way; and any considerable errors in the methods

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tion of 2½ *per cent.* was in consequence made to the claims, so that at present the assurances made in 1800 are increased £27 *per cent.*, those in 1790 £86 *per cent.*, those in 1780 £120 *per cent.*, those in 1770 £290 *per cent.*, and those in 1767 £323 *per cent.*; and a sum equivalent in present money to one *million* is thus distributed among the members in addition to the sums which have been distributed on former occasions. It is unnecessary to expatiate on this subject, in order to shew the great benefit which this Society has proved to the public.—Proceeding always as it has hitherto done on the sure ground of computation, it never can incur any material danger.—The many thousands of which it is now composed render the decrements of life almost uniform among its members, and from the experience of more than forty years those decrements are found to be even *one third* (*a*) lower than they are in the Table from which the premiums are computed. Provided therefore the same care is used in excluding bad lives, every addition to the business of the Society must be an addition to its profits.—The increased number of members, by preventing it from being affected by any particular season of mortality, will also render those profits more uniform and secure; so that the stability of the Society will be strengthened in proportion as its usefulness is extended.

The only danger to which it is exposed arises from the greatness of its capital. If due regard be not paid to the immense demands which must hereafter come upon it.—If on the contrary this capital be considered only as an accu-

(*a*) See note p. 183.

methods of solving such questions, must in time produce very bad consequences.—The design of the following observations is to point out a particular error, into which there is danger of falling, in finding the values of such reversions as depend on survivorships. In doing this, I shall, in order to be as plain as possible, take the following case. "A, "aged 40, expects to come to the possession "of an estate, should he survive B, aged "likewise 40. In these circumstances he "offers, in order to raise a present sum, to "give security for £40 *per annum*, out of "the estate at his death, provided he should "get into possession; that is, provided he

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accumulation of profit, and the members on this supposition should be tempted to make too frequent additions to the claims, the consequence, though distant, must ultimately be fatal. Hitherto great prudence and discretion have been manifested in the management of this Society. No measure has been adopted which had the least tendency to diminish the capital without a previous investigation; and in order to secure the same prudence and discretion in the future conduct of its members, the Society has provided by certain laws, "that no allowance "to claimants shall ever be made without a previous in-"vestigation of its affairs and the concurrence of *four-"fifths* of its members at three successive General "Courts, and also that such allowance shall in no in-"stance exceed *two-thirds* of the clear surplus stock of "the Society." By these restrictions, as far as human precaution can operate, the danger of intemperate measures is avoided; and there is every reason to hope that an institution, founded on such liberal principles and defended by such wholesome provisions, will not only be secure, but continue to improve in credit and prosperity.

M.

"should survive B. What is the sum that "ought now to be advanced to him, in consideration of such security, reckoning compound interest at 4 *per cent*?"

Mr. *De Moivre's* directions in his Treatise on Annuities, Problems 17th and 20th, lead us to seek the required sum in this case, by the following process.

Find first, the present sum A should receive, for the reversion of £40 *per annum* for ever after his death; supposing it *not* dependent on his surviving B. The present value of such a reversion is "the[1] value of the "life subtracted from the *perpetuity*, and "the *remainder* multiplied by the annual "rent."—The value of the life is, by Mr. *De Moivre's* Hypothesis, 13.196. This subtracted from 25, the *perpetuity*, leaves 11.804; which, multiplied by 40, gives £472.16; the value of the supposed estate, after the life of A. But, as Mr. *De Moivre* observes, the lender having a chance to lose his money, a compensation ought to be made to him for the risk he runs, which is founded on the possibility, that a man of 40 years of age may not survive another person of the same age. This chance is an *equal* chance; and, therefore, half the preceding sum, or £236.08, is the money which should be advanced now on the expectation mentioned.

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[1] By *Scholium*, p. 34, and Problem 26th, p. 293 of Mr. *Simpson's* Select Exercises.

This solution carries a plausible appearance; and most persons will, probably, be ready to pronounce it right; nor will this be at all wonderful, as so great a master of these subjects as Mr. *De Moivre*, appears to have been misled by it.—Nothing more is necessary to prove it to be fallacious, than proceeding in the same way to solve the following similar Question.

" A, aged 40, offers to give security for
" £40 *per annum*, to be entered upon at his
" death, provided it should happen *before* the
" death of B, aged likewise 40. What sum
" should now be advanced to him for such a
" reversion, interest being reckoned at 4
" *per cent?*"

In solving this problem, agreeably to the method just described, we are to find the value of £40 *per annum*, to be entered upon *certainly* at the death of A; and then to multiply this value by the chance that A shall *not* survive B, or by $\frac{1}{2}$; and in this way the answer comes out the same with that already given.

Now it may be easily seen, that this must be wrong. The value of a reversion, to be received when a person of a given age dies, cannot be the same, whether the condition of obtaining it is, that he shall die *before*, or that he shall die *after* another person. That is, whether it is provided, that a purchaser, if he succeeds, shall get into possession *sooner* or *later*. The reversion in the latter case must, without doubt, be of less value than in the former.

The first question here proposed, resolves itself into the following general Question.

"What is the present value of a given reversionary estate, to be entered upon after "the failure of two lives, provided one *in* "*particular* of them should be the *longest* "*life*?"

Now, the present value of an estate to be enjoyed for ever, after the failure of the *longest* of two lives, is "the value of the *longest* "of the two lives, subtracted from the *per*-"*petuity*; and the remainder multiplied by "the annual rent of the estate."—The value of the *longest* of two lives is, as is well known, the value of the two joint lives, subtracted from the sum of the[m] values of the two *single* lives. In the present case, therefore, it is 9.82, (the value of two joint lives at the age of 40 by Table VII. in the next volume) subtracted from twice 13.196; (the value of a *single* life at the same age) that is, 16.57 years' purchase. And this subtracted from 25, (the perpetuity) gives 8.43; which, multiplied by 40, gives £337.2, the value of the given estate were it *certainly* to be enjoyed, after the extinction of the longest of two lives both 40; that is, whether *one* or *other* of them failed last. But that A's life in particular should fail last, is an even chance. The true value of the reversion, therefore, is half the last value, or £168.6.

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[m] See Mr. *De Moivre* on Annuities, Problem IV. or Mr. *Simpson's Doctrine of Annuities and Reversions*, Problem II.

In like manner. The second Question is the same with the Question, "What is the "present value of £40 *per ann.* for ever to "be entered upon after the extinction of two "*joint* lives both 40; that is, whenever *either* "of them shall fail; provided the first that "fails should happen to be A's life in par"ticular?"—And the answer is found by subtracting the present value of the *two joint* lives from the *perpetuity,* and multiplying the remainder by ½, that is, by the chance that A in particular shall die first: And this will give the required value, £303.6[n].

In short. It appears in *both* these cases, that, according to the first method of solution, we are to subtract from the *perpetuity* the value of *one* of the single lives, when, in the *former* case, the value of the *longest* of the two lives, and, in the *latter* case, the value of their *joint continuance,* ought, in reality, to be subtracted. I need not say what prodigious errors may often arise from hence; and how unfit such a method of solution is for practice.

Mr. *Simpson,* in p. 322, of his Select Exercises, speaks on this subject in the following manner.—" I have been very particular " on these kinds of Problems; and the more " so, as there has been no method before " published, that I know of, by which they " can be rightly determined. 'Tis true, the " manner of proceeding, by first finding the " probability of Survivorship (which me- " thod is used in my former work, and " which a celebrated author has largely in- " sisted on in three successive editions), may " be applied to good advantage, when the " given ages are nearly equal; but then it is " certain, that this is not a genuine way of " going to work, and that the conclusions " hence derived are at best but near approxi- " mations."

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[n] I have, though scarcely necessary, given a demonstration of the Solutions in Note M. at the end of this volume.

This excellent mathematician has here expressed himself much too favourably of the method of solution on which I have remarked.—In both the cases I have specified, the ages are equal; and yet, in one of them the error is a good deal above a *third* of the true value, and in the other a *fifth*: And, it is obvious, that in cases where three equal lives are taken, the errors will be much greater. —Mr. *Simpson's* Observations in this passage are true only, when applied to a *different* method used by himself, in the 28th and following Problems of his Treatise on the *Doctrine* of *Annuities* and *Reversions*. This method is exact when the lives are equal; but it gives results which are too far from the truth, when there is any considerable inequality between the lives.

It is with reluctance I have made some of these remarks. Mr. *De Moivre* has made very important improvements in this branch of science; and the highest respect is due to his name and authority. This, however, only renders these remarks more necessary.

In the first Chapter (Question 11th, 12th, 15th, and 16th °.) I have given a minute account of the method of finding, in all cases, the values of the reversions which have been the subject of this Essay.—But Mr. *Morgan*, in his Treatise on Life Assurances, has carried this enquiry much farther.

° In solving Questions 11th and 12th, here referred to, I have made use of Mr. *Simpson's* rules in his Select Exercises, p. 297 and p. 298. Mr. *Morgan*, in his Treatise on Life Assurances, adopted the method of solution from which Mr. *Simpson* deduced these rules; but he has lately discovered that it is an approximation which in some cases gives results that are much too distant from the truth. See more on this subject at the end of this volume, where rules (investigated by Mr. *Morgan*) may be found which give in all cases the correct values of sums payable on any survivorships between any two or three lives.

Account of the Values of the Renewal of Leases, and of the Method of computing them.

TABLE,

Shewing the Fines due on the Renewal of a Lease of 21 Years after 5, 7, 9, or 11 Years have elapsed.

Years unexpired.	Values of the Renewal.							
	At 4 per cent.		At 6 per cent		At 8 per cent.		At 10 per cent	
16	$2\frac{3}{8}$	Years purchase	$1\frac{2}{3}$	Years purchase	$1\frac{1}{6}$	Years purchase	$\frac{4}{5}$	Years purchase
14	$3\frac{1}{2}$		$2\frac{1}{2}$		$1\frac{3}{4}$		$1\frac{3}{10}$	
12	$4\frac{2}{3}$		$3\frac{3}{8}$		$2\frac{1}{2}$		$1\frac{5}{6}$	
10	$5\frac{9}{10}$		$4\frac{2}{5}$		$3\frac{3}{10}$		$2\frac{1}{2}$	

The value in every case of this kind is the difference between the value (in Table II. Vol. II.) of the whole term, and the value (in the same Table) of the unexpired part of the term.

If leases are held by lives, the value of their renewal is the difference between the value of all the lives (including the life or lives to be added) and the value of the existing life or lives.——For example.

The value of the renewal of a lease held by two lives after one has dropped is (supposing the existing life aged 50, and the life to be added aged 20) the *difference* between

16.033

16.033 (the value by Table XIX. of the single life of 20) and 9.630 the value by Table XX. of the joint lives 20 and 50 [p]. That is, 7.403, or $7\frac{2}{5}$ of a year's purchase nearly, reckoning interest at 4 *per cent.*——Again, the value of the renewal of a lease held by *three* lives, after *one* is dropped, is (supposing the two existing lives aged 50 and 55, and the life to be added aged 20) the *difference* between the value of the longest of the three lives and the value of the longest of two lives aged 50 and 55. The former of these values by Tables XIX. and XX. is 18.178 [q], the latter by the same Tables is 13.872 [r]; their difference is 4.306, or $4\frac{3}{10}$ years purchase; which, therefore, is the fine due for such a renewal, reckoning interest at 4 *per cent.*

N.B. If the values of such renewals are wanted at any rates of interest higher or lower than those for which the values of single and joint lives are given in the Tables in the 2d volume, they must be deduced from the values in those Tables by the Rules given in the next Chapter.

It would be an endless labour to compute tables shewing the value of such renewals in all cases; and these directions render it an unnecessary labour.

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[p] See page 2.

[q] By Prob. X. in *Simpson's* Select Exercises, or by the Rule at the end of Table XXVI. Vol. II.

[r] See the note in page 43 of this volume.

Sometimes a right may be purchased to put in, on the first vacancy among the lives by which an estate is held, such a new life as the purchaser shall chuse.—In order to find the present value of such a right, it is necessary to assume some given value for the life to be nominated, and this assumed value multiplied by the difference between the value of the existing life, if there is but one (or the value of the joint continuance of the existing lives, if there are two or more) and the perpetuity; and the product, divided by the perpetuity, will give the answer.

EXAMPLE.

Let there be but one existing life, and let it be a male life, its age 50, and consequently its value (by Table XIX.) 10.269, reckoning interest at 5 *per cent.*——Let the life to succeed it be reckoned a life of the greatest possible value, that is, a life aged 8, and consequently worth (by the same Table) 15.226 year's purchase at 5 *per cent.*—The difference between 20 (the perpetuity) and 10.269 multiplied by 15.226 is 148.164; which product, divided by 20, gives 7.408, the answer.

If there are two existing lives, both 50, the value of their joint continuance will be (by Table XX.) 7.522; the difference between which value and the perpetuity is 12.478, which multiplied by 15.226, and the product divided by the perpetuity, gives 9.5 the answer in this case, or the number of years purchase which ought to be paid for a right of renewing a lease now held by two lives both aged 50, by putting in the best life in the room of the first of the two lives that shall happen to drop.

The rule for finding the value is the same, if the right to be sold is the right of presentation to a church-living at the death of the present incumbent.

The estate meant in these rules is the nett *surplus* rent after deducting all taxes and repairs.

CHAPTER IV.

On Mr. De Moivre's *Rules for calculating the Values of* Joint *Lives; with a* Post-script, *containing a Specimen of the most expeditious Method of calculating the Values of Single and Joint Lives, according to any Table of Observations.*

THE calculation of the values of *single* and *joint* lives, from given Tables of Observation, being tedious and troublesome; Mr. *De Moivre* has had recourse to two *hypotheses*, which give easy rules for this purpose; and which, he thought, corresponded with sufficient exactness to Observation.—The first of these *hypotheses* is, that the probabilities of life decrease, as we advance from childhood to old age, in an *arithmetical progression*; or in such a manner, that the *difference* is always the same, between the number of persons living at the beginning of any one year, and the number living at the beginning of the next following year.—The other *hypothesis* is, that the probabilities of life decrease in a *geometrical* progression; or in such manner, that the *proportion* is always

ways the same, between the number of persons living at the beginning of any one year, and the number living at the beginning of the next following year.—All the Tables of Observation shew, that the real law, according to which human life wastes, comes much nearer to the former *hypothesis*, than the latter.—In Tables V. VIII. and XVII. in the next volume, it is so near the former *hypothesis*, that the difference between them in the middle stages of life is scarcely worth regarding. According to this *hypothesis*, therefore (accommodated to the *Breslaw* Table, in the manner mentioned in the note, page 2.) Mr. *De Moivre* calculated the values of *single lives*; and the rules founded upon it for this purpose are so easy, that an operation which would otherwise take up much time, may be performed almost immediately.

By proceeding on the same principles, the values of *joint* lives might have been calculated; but the rules for this purpose derived from these principles, are far from being equally easy in practice. Here, therefore, Mr. *De Moivre* quitted his *first* hypothesis; and finding, that the *second* hypothesis afforded, in the case of *joint* lives, rules that were as easy as the rules given by the other hypothesis were in the case of *single* lives, he chose to adopt this *hypothesis*; believing at the same time, that the values of *joint* lives, obtained by rules derived from it, would not deviate much from the truth. But in this he was greatly mistaken. The values of *two joint* lives obtained by these rules are so wrong, that in finding the present value, in a *single payment*, of one life after another, they generally give results, which are near a *quarter* of the true value too great; and about *two-fifths* too great, when the value is sought in *annual payments* during the joint lives. These are errors so considerable, that I think it of particular importance that the public should be informed of them, in order to prevent the inconveniencies and perplexities they may occasion.

Mr. *Simpson* (in the Appendix to his Treatise on the *Doctrine of Annuities and Reversions*) has observed, that Mr. *De Moivre's* rules for finding the values of joint lives are wrong. But I don't know, that it has been ever attended to, that they are so wrong as I have found them. Mr. *Simpson's* remarks point out chiefly the errors in these rules, when the values of *three* or more joint lives are calculated by them; but, till I was forced to a particular examination of this subject by some difficulties into which I found myself brought by following Mr. *De Moivre* too implicitly, I did not at all suspect, that any such errors as I have mentioned could arise from these rules, when the values of only *two* joint lives are calculated by them. Mr. *De Moivre*, in consequence of other remarks contained in Mr. *Simpson's Appendix*, altered, in the 4th edition

tion of his Treatise, some of his rules. It is surprizing he did not see reason at the same time to alter these.

That there may be no doubt about the truth of these observations, I will just mention a few examples of the difference between the values of a given reversionary annuity, according to the rules to which I have objected, and the values, according to the exact method of deducing them from Mr. *De Moivre's first hypothesis.*

Let the proposed annuity be £30 to be enjoyed for what shall happen to remain of the life of a person now 40 years of age, after the life of another person of the same age. The value of the joint lives (interest being at 4 *per cent.*) is, by the 2d hypothesis, or Problem 2d of Mr. *De Moivre's* Treatise on Life-Annuities, 8.964; which subtracted from 13.196, (the value, by the first hypothesis, of a single life at 40) gives 4.23; which remainder, multiplied by 30, gives £126.9, or the value of the reversion in a single present payment. And 126.9, divided by the foregoing value of the joint lives, is £14.16; or, the value of the reversion in annual payments, during the joint lives.—But the *true* values are £101.1 in a *single* payment, by Quest. I. Chap. I. and £10.3, in *annual* payments, by Quest. IV. —The former values, therefore, are a *quarter* of the true value too great in the *single* payment;

ment; and near *two-fifths* too great in the *annual* payments.

The *true* value of the same annuity for a life at 66, after another life of the same age, is, (reckoning interest as before, at 4 *per cent.*) £68 in a *single* payment; and £13.5 in *annual* payments.—But these values, according to the Problem just quoted, are £91 and £21, one of which is near a *third*, and the other above *half* the true value too great.

In *unequal* lives these errors may be no less considerable.—Thus; if the value of the proposed annuity be required for a life at 70, after a life at 20 years of age; it will, by the same Problem, be £26.5, in a *single* payment; and £5.1, in *annual* payments during the joint lives. But the *true* values are £17 and £3.05.

Where 3 or more lives are concerned the errors will be still greater.

The true values of the joint lives, mentioned in these Examples, have been calculated by a rule in page 16, of Mr. *Simpson's* Treatise on the Doctrine of *Annuities* and *Reversions*, and explained in Note (L) at the end of this volume.—To save, however, a great deal of trouble hereafter, I have thought proper to calculate Table VII. in the next volume, which gives the exact values according to Mr. *De Moivre's first* hypothesis, of two joint lives, for every five years of human life, from 10 to 70.

This *hypothesis*, I have observed, does not differ much from the Tables of Observation for *Breslaw*, *Northampton*, and *Norwich*. Between the ages of 30 and 40, it gives the values of *single* lives almost the same with the *Breslaw* Table. Under 30, it gives them somewhat less; and above 40, somewhat *greater*. But it ought to be remembered, that whereever it does this, it gives, at the same ages, the values of the *joint* lives also too little or too great; and that, consequently, the results from it, in calculating the values of *Reversions*, and of the *longest* of given lives, come so much nearer to exactness.

The rules to which I have objected are the only ones given by Mr. *De Moivre*, in all the editions of his Treatise on Life-Annuities. But it seems, this great mathematician became at last sensible, that they were too incorrect; and therefore, at the end of the last edition of his Treatise on the *Doctrine of Chances*, page 320, (a work which gets into comparatively few hands) he has given other rules which come nearer the truth. But even these rules produce errors so great in many cases, (particularly when combined with the errors of the hypothesis) that it will be best never to use them.

POSTSCRIPT

FOR THE FOURTH EDITION.

SINCE the former editions of this work I have found reason to be dissatisfied with Mr. *De Moivre's first* as well as his *second* hypothesis. There is no situation in which, in the first and last periods of life, it corresponds to fact; and in some situations, particularly great towns and country parishes, it does not correspond sufficiently to fact in *any* periods of life. An inspection of the Tables of Observation in the next volume will prove this. However useful, therefore, this hypothesis may be in many cases, it would be best not to be under any necessity of having recourse to it; and for this reason, and also to render this work as complete as I am capable of making it, I have, while this edition has been in the press, and with the help of some friends, calculated the tables in the next volume of the values of single and joint [s] lives from the *Northampton* register of mortality. This register has been chosen for this purpose, because it gives the mean values of lives between the highest and lowest, and is on this account, and also in consequence of the corrections I have made in it, better fitted for general use than any other.——I have, however, retained the Tables of these values according to Mr. *De Moivre's* Hypothesis, published in the former editions of this work; because all the examples in the preceding part of this work have been taken from them, and there are some cases in which they may still prove of use.

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[s] The value of the reversionary annuity, mentioned in p. 208, is by these Tables 14.83 in annual payments, instead of 13.5, as there given from the hypothesis.

The computation of the values of joint lives correctly from a given table of Observations, is a business so tedious and tiresome, that it has scarcely been ever executed, except by Mr. *Simpson* from the *London* Observations; and as these give the values of lives among a body of people taken in the gross in one of the worst of all situations, they are by no means fit for common use.——I have therefore, employed a good deal of attention to find out the most easy and expeditious method of making these calculations; and I shall here give the following *Specimen* of a method (deduced from that described by Mr. *Morgan* in his Treatise on Life-Annuities and Assurances, chap. 2d, sect. 2d, p. 56) which, at the same time that it renders mistakes impossible, will expedite this work as much as the nature of it will allow, and render the computation of the values of any number of *joint* lives not more difficult or tedious than the computation of the values of an equal number of *single* lives.

Let the Table of Observations be that for *Northampton*, or Table 17th, in the next volume; and let the rate of interest be 4 *per cent.*

Write down on a paper to be always kept in sight the Logarithms of all the numbers in the column of the living without the Indices.

EXAMPLE FIRST.

Living at age 0—	11650—	Log^{ms}. .066325
1 year	8650—	.937016
&c.	&c.	&c.
age 81—	406—	.608526
82—	346—	.539076
83—	289—	.460897
84—	234—	.369215
&c.	&c.	&c.
age 91—	34—	.531478
92—	24—	.380211
93—	16—	.204119
94—	9—	.954242
95—	4—	.602059
96—	1—	.000000

Find the Logarithm of £1, increased by its interest for a year, and also the Logarithm of the value of £1, payable at the end of a number of years equal to the difference between the greatest and least ages in the Table of Observations lessened by the difference of age between the joint lives whose values are to be calculated.

EXAMPLE SECOND.

Interest being at 4 *per cent.* £1, increased by its interest for a year is 1.04; and the Logarithm of 1.04 is .0170333.

In the *Northampton* Table of Observations the greatest age is 96, and the least age 0. The difference, therefore, is 96; and supposing the given difference of age between the two joint lives to be 10 years, the value of £1 payable at the end of a number of years equal to the difference between the greatest and least ages in the table lessened by the difference of age between the joint lives, will be the value of £1 payable at the end of 86 years. Table 1st, in the next volume, shews this value to be .0342872 (reckoning interest at 4 *per cent.*) the Logarithm of which number (striking out the Index) is .535133.

N. B. The best way of finding this Logarithm is by multiplying the Logarithm of £1 with its interest for a year by the difference between the greatest and least ages in the Table lessened by the difference of age between the joint lives, and subtracting the product from unity. The remainder will be the Logarithm sought. Thus, in the present example .0170333 multiplied by 86, gives (without the Index) .464864, which subtracted from unity leaves .535136.——Had the given difference of age between the two joint lives been 15 years, and the youngest age in the Table of Observations 3, and the oldest 94, the Logarithm .0170333, instead of being multiplied by 86, must have been multiplied by 76, and the product (without the Index) subtracted from unity would have been .705469.

Having made these preparations, the calculations must begin with the oldest joint lives, and proceed upwards according to the following specimen.

SPECIMEN of an easy and expeditious Method of calculating the Values of two Joint Lives,

Interest at 4 *per Cent.*——Difference of Age 10 Years.——*Northampton* Table of Observations.

Values — — — — Ages $\left.\begin{matrix}95\\85\end{matrix}\right\}$ .18739—Ages $\left.\begin{matrix}94\\84\end{matrix}\right\}$ .403347—Ages $\left.\begin{matrix}93\\83\end{matrix}\right\}$ .61455—Ages $\left.\begin{matrix}92\\82\end{matrix}\right\}$ .86416—Ages $\left.\begin{matrix}91\\81\end{matrix}\right\}$ 1.07846

.017033 (*a*) added continually to .535136 See *second* Example	A	.535136	.552169	.569202	.586235	.603268	
Log^ms^. of the numbers living at 96, 95, 94, &c. years. See *first* Example	B	.000000	.602059	.954242	.204119	.380211	.531478
Log^ms^. of the numbers living at 86, 85, 84, &c. years. See *first* Example	C	.161368	.269512	.369215	.460897	.5 39076	.608526
B + C	D	.161368	.871571	.323457	.665016	.919287	.140004
Log^ms^. of the numbers (with unity added) of 1st, 2d, 3d, &c. Log^ms^. in H	E	......	.074597	.147162	.208051	.270553	
D + E	F	.161368	.916168	.470619	.873067	.189840	
.017033 (*a*) added to 2d, 3d, 4th, &c. Logarithms in D	G	.888604	.340490	.682019	.936320	.157037	
F − G	H	.272764	.605678	.788570	.936747	.032803	
A + G	I	.423740	.892659	.251251	.522555	.760305	
H + I	K	.696504	.498337	.039821	.459302	.793108	
Numbers of the Log^ms^. in I	L	.265303	.781016	.178341	.333087	.575846	
Numbers of the Log^ms^. in K	M	49717	.315019	.109603	.287941	.621025	
Proof or L + M		315020	1096035	.287944	.621028		

(*a*) This must be .0128372—.0149103—.017033—.021189—.0253059, as the rates of interest are 3, 3½, 4, 5, or 6 *per cent.* and the first Logarithm in A for these several rates of interest, on which all that follow in that line depend, is to be deduced from them by the rule in the *second* Example.

OBSERVATIONS.

In the addition of the Logarithms in this Specimen, the decimal parts only are to be retained.

In subtracting them, it is of no consequence whether a Logarithm is greater or less than that from which it is to be subtracted.

In every column, the numbers in the lines B, C. D, E, F, G, H, give the *value*. The other numbers give the *proof*.

The *first* Logarithms in the lines B and C are always the Logarithms of the numbers of the living at the oldest ages in the Table of Observations, which have the given difference of age, and the following Logarithms, are the Logarithms of the numbers living at the next ages, each one year younger than the preceding.

The values of the two joint lives are the numbers of the Logarithms in H; and the proof of these values consists in the equality of the sum of the numbers in L and M in one column, to the number in M in the following column. And it should be particularly observed that this proof answers sufficiently if, in consequence of placing the numbers in L and M over one another in *any* order, a sum can be made out whose first 6 figures are equal within 5 or 6 units to the succeeding number in M. If the proof does not answer within this limit, the calculations have been too incorrect[1], and it will be necessary to examine the numbers not verified by the last proof; namely, the numbers of the two Logarithms in I and H in the *preceding* column; the Logarithms in B and C and D in the *subsequent* column; and the Logarithms in A, E, F, G, H, I, and K, in the column where the proof is found to be deficient.

EXAMPLE.

The addition, in the foregoing specimen, of the first numbers in L and M to one another, when the *first* figure in M is placed under the *second* in L, makes 315020, which is within one unit the same with 315019, the second number in the line M; and this proves the calculations so far to be sufficiently correct.——In like manner, the addition of the *third* numbers in L and M gives the *fourth* in M within three units. But had the addition of the numbers in L and M given the subsequent number in M only within *six* units; that is, had it given the number 287947 or 287935, an incorrectness of too much consequence must have insinuated itself, and it would have been proper to examine the numbers and Logarithms just mentioned, in order to detect it.

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[1] This supposes that the values of the joint lives are to be found to three places of decimals. The agreement of *five* figures within 4 or 5 units will be sufficient, if the values are required only to two places of decimals.

In calculating the *last* value; that is, the value when the youngest of the lives is the youngest in the Table (or a life just born, according to most Tables) this proof will change into a new proof, verifying all the preceding values. For the Logarithm of £1 with its interest for a year (that is, .017033 when the interest is 4 *per cent.*) subtracted from the Logarithm in F, will leave a Logarithm, the number of which will be the sum of the numbers in L and M in the preceding column.

This Specimen will be accommodated to the calculation of the values of *single* lives[u], by striking out the Logarithms in C, and making those in D the same with those in B; and also making the first Logarithm in A, from which all that follow are deduced, the Logarithm of £1 payable at the end of a number of years, equal to the oldest age in the Table when it begins at birth, or to the difference between the oldest and youngest when it begins at any age after birth.

This Specimen will be accommodated to the calculation of the values of any *three* joint lives, if to the Logarithms in B and C are added (in order to obtain D) the Logarithms of the numbers living at any other ages, the difference between which and the ages in B and C is given; and if, likewise, the series of Logarithms in A is deduced from the Logarithm of the value of £1, payable at the end of a number of years equal (if the ages begin at 0) to the oldest in the Table; or (if they do not begin at 0) to the difference of age between the oldest and youngest in the Table; lessened (in both cases) by the differences of age between the oldest and youngest of the three joint lives whose values are sought.——Thus. Supposing the given differences between the ages of the joint lives whose values are to be calculated to be 5 and 10 years, and the interest 4 *per cent.* and the Table of Observations to terminate (as the *Northampton* Table does) at 96 years of age, and to begin at 0. The series of Logarithms to be added to those in B and C in order to obtain D, will be the Logarithms of the living at 91 in the first column, at 90 in the second, at 89 in the third, &c. And the Logarithms in A will be the same with those in the Specimen. But had the differences of age been 10 and 15 years, the Logarithms to be added to those in B and C would have been the Logarithms of the living at 81, 80, 79, &c. and the first Logarithm in A would have been the product of 81 into .017033 subtracted from unity or .620302, and the following Logarithms in A would have been

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[u] In this case, and also in calculating the values of *equal* joint lives, the first Logarithm in F (when the number living at the oldest age in the Table is 1) will be nothing; but notwithstanding this, the first Logarithm in G must be subtracted from it just as if it was unity, in order to obtain the first Logarithm in H.

.017033

.017033 added continually to this Logarithm.

It is hence evident that in this method computations of the values of any given *two* or *three* or *four* joint lives are nearly as easy as computations of the values of *single* lives; and may, after some practice, be performed almost as expeditiously as the numbers can be written.

An error in a book of Logarithms may, if not suspected, produce infinite perplexity; and therefore, when, after repeating any calculation, the source of an error cannot be discovered, it will be right to examine the Tables from whence the Logarithms have been taken. In general, the order in which the numbers follow one another will immediately discover an error of the press; but if not, a different book of Logarithms should be consulted; and if possible, Mr. GARDINER's, which is so correct as to be almost invaluable.

It may be proper to observe once more, that it is very easy to take from SHERWIN's or GARDINER's Tables the numbers of Logarithms, and the Logarithms of numbers to *six* figures; and that if this is done, the resulting values will be always accurate to at least the *third* place of decimals. But, if such a degree of accuracy is thought needless, it will be sufficient to take them to *five* figures.

The Theorems on which the method of calculations here explained (exclusive of the proof) are grounded, will be given in note N at the end of this volume. But a more distinct investigation of these Theorems, and also an explanation of the principles on which the *proof* is founded, has been given by Mr. *Morgan* in his Treatise on Annuities and Assurances. Chap. 2d. Sect. 2d.

The rules for finding, from the values of two or three joint lives, the values of the *longest* of any *two* or *three* lives; and also a very easy rule for obtaining *nearly* the value of any *three* joint lives from the values of *two* joint lives, will be given in the next volume at the end of the Table, shewing the values of two joint lives according to the *Northampton* Observations.

Account of a method of deducing, from the correct *values (according to any observations) of any single or joint lives at* one *rate of interest, the same values at* other *rates of interest.*

PRELIMINARY PROBLEMS.

Problem I. The *expectation* given of a single life by any table of observations, to find its value, supposing the decrements of life equal, at any given rate of interest.

Solution. Finding in Table II. the value of an annuity certain for a number of years equal to twice the expectation. Multiply this value by the Perpetuity increased by unity, and divide the product by twice the expectation. The quotient subtracted from the perpetuity will be the value required.

EXAMPLE.

The *expectation* of a life aged ten, by the *Northampton* observations (See Table XVIII.) is 39.78. Twice this expectation is 79.56. The value of an annuity certain for 79.56 years is, by Table II. (reckoning interest at 4 *per cent.*) 23.894. The product of 23.894 into 26 (the perpetuity increased by unity) is 621.244, which, divided by 19.56, gives 7.808. And this quotient subtracted from 25 (the perpetuity) gives 17.192 years purchase, the value of a life aged ten, deduced from the expectation of life at that age, according to the *Northampton* observations.

This is the rule by which Mr. *De Moivre* has calculated the table commonly used of the values of lives according to his hypothesis; and from this Table (the sixth Table in the next volume) the value required in this problem may be deduced more compendiously in the following manner, provided the expectation does not exceed 38.— "Take the difference between twice the

“ expectation and 86; and the value in the “ Table corresponding to that difference, if “ not less than 10, will be the value sought.” Thus; twice the *expectation* of a life aged 30 (that is, its *complement*) is, by Table XVIII. 56.54. The difference between it and 86, is 29.46. And since the value corresponding to age 29 in Mr. *De Moivre's* valuation of lives (or in Table VI. in the next volume) is (reckoning interest at 4 *per cent.*) 14.816; and the value corresponding to age 30 is 14.614; it is obvious, that the value corresponding to age 29.46 must be the *greatest* of these two values lessened by $\frac{46}{100}$ of the difference between it and the *least*. This difference is 202; and $\frac{46}{100}$ of it (or .202 multiplied by .46) is .093, which subtracted from 14.816, gives 14.723 the value sought of a life whose expectation is 28.27 (or whose complement is 56.54) on the supposition of an equal decrement of life.

PROBLEM II.

Having the expectations given of any two lives by any table of observations, to deduce from thence the value of the joint lives at any rate of interest supposing an equal decrement of life.

Solution. Find the difference between twice the expectation of the youngest life, and twice the expectation of the oldest life increased by unity and twice the perpetuity.

Multiply this difference by the value of an annuity certain for a time equal to *twice* the expectation of the *oldest* life; and by twice the same expectation divide the product, reserving the quotient.

From twice the perpetuity subtract the reserved quotient, and multiply the remainder by the perpetuity increased by unity [x]. This last product divided by twice the expectation of the *youngest* life, and then subtracted from the perpetuity, will be the required value.

EXAMPLE.

Let the joint lives proposed be a life aged 10, and another aged 15, and let the table of observations be the *Northampton* Table, and the rate of interest 4 *per cent.* Twice the expectations of the two lives are 79.56 and 73.02. (See Table XVIII.)

Twice the expectation of the *oldest* life, increased by unity and twice the perpetuity, is 124.02, which lessened by 79.56 (twice the expectation of the *youngest* life) leaves 44.46 for the *reserved* remainder.——This remainder multiplied by 23.574 (the value of an annuity certain[y] for 73.02 years) and the product divided by 73.02 (twice the expectation of the *oldest* life) gives 14.352 the quotient to be *reserved;* which subtracted from double the perpetuity, and the remainder (or 35.648) multiplied by the perpetuity increased by unity (or by 26) gives 926 85, which divided by 79.56 (twice the expectation of the *youngest* life) and the quotient subtracted from the perpetuity, we have 13.350 for the required value.

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[x] When twice the expectation of the youngest life is *greater* than twice the expectation of the oldest life increased by unity and twice the perpetuity, the reserved quotient, instead of being *subtracted* from twice the perpetuity, must be added to it, and the *sum*, not the *difference*, multiplied by the perpetuity increased by unity.

This calculation may be made more easily by logarithms in the following manner.

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[y] This value, when the number of years is a whole number with a fraction added (as will be commonly the case) may be best computed in the following manner.

Suppose the number of years to be 83.28.

The value of an annuity certain for 83 years is (by Table II. Vol. II.) 24.035. The same value for 84 years is 24.072. The difference between these two values is .037; which difference multiplied by .28 (the fractional part of the number of years) and the product (.0103) added to the *least* of the two values, will give 24.045 the value for 83 28 years.

Twice the expectation of the eldest is	73.02	
ADD twice the perpetuity increased by unity - -	51.00	
Sum - -	124.02	
SUBTRACT twice the expectation of the youngest -	79.56	
Remainder -	44.46	
Log. of 44.46 is	1.647,9695	
Log. of 23.574 is	1.372,4333	
Sum - -	3.020,4028	
Subtract -	1.863,4418	Log. of 73.02
Remainder -	1.156,9610	the number of which is 14.352 or the remainder to be reserved.

Twice the perpetuity is	50	
Subtract	14.352	
Remains - -	35.648 Logar. of which is	1.552,0352
Add Log. of the perpetuity increased by unity		1.414,9733
Sum - -		2.967,0085
Subtract Log. of twice the expectation of the youngest - - - - - - - - - - -		1.900,6948
Remainder - -		1.066,3137

The number of this last remainder is 11.650, which subtracted from 25 (the perpetuity) leaves 13.350, the value sought.—See the algebraical canon in Note (L) at the end of this volume.

GENERAL

GENERAL RULE.

Call the *correct* value (supposed to be computed for any rate of interest) the FIRST value.

Call the value deduced (by the preceding problems) from the *expectations* at the same rate of interest, the SECOND value.

Call the value deduced from the *expectations* for any other rate of interest the THIRD value.

Then, the difference between the *first* and *second* values added *to* or subtracted *from* the *third* value, just as the *first* is *greater* or *less* than the *second*, will be the value at the rate of interest for which the THIRD value has been deduced from the expectations.

The following examples will make this perfectly plain.

EXAMPLE I.

In the 48th and 49th Tables, Vol. II. the correct values are given of two joint lives among mankind at large, without distinguishing between males and females, according to the SWEDEN observations, reckoning interest at 4 *per cent.*

Let it be required to find from these values the values at 3 *per cent.*; and let the ages of the joint lives be supposed 10 and 10.

The correct value by Table XLVIII. (reckoning interest at 4 *per cent.*) is 16.141. The expectation of a life aged 10 is (by Table XLIV.) 45.07. The value deduced from this expectation at 4 *per cent.* by Prob. II. is 14.539. The value deduced by the same problem from the same expectation at 3 *per cent.* is 16.808. The difference between the *first* and *second* values, is 1.602, which, added to the *third* value (the first being *greater* than the second) makes 18.410 the value required.

EXAMPE II.

Let the values be required of two joint lives aged 50 and 60, at an interest of 3 *per cent.* from the correct value given at an interest of 4 *per cent.* according to the *Northampton* observations.

First or correct value at 4 *per cent.* by Table XXII. is 6.989. The expectation of 50 is 17.99; of 60, is 13.21, by Table XVIII. The *second* value, or the value deduced from these expectations at 4 *per cent.* is, by Prob. II. 7.182. The *third* value, or the value deduced from the same expectations at 3 *per cent.* is 7.704.—The difference between the first and second is .193, which (since the *second* is greater than the *first*) must be subtracted from the *third*, and the remainder (or 7.511) will be the value required.—The exact value at 3 *per cent.* is, by Table XXII. 7.460.

If the value is required at 5 *per cent.* the *third* value will be 6.732; and the difference subtracted from 6.732, will leave 6.539 the value at 5 *per cent.*

The *exact* value at this rate of interest, is (by Table XXII.) 6.568.

EXAMPLE III.

Let the value be required of a single male life aged 10, at 3 *per cent.* interest, from the correct value at 4 *per cent.* according to the *Sweden* observations.

First, or correct value at 4 *per cent.* (by Table XLVII.) is 18.674. The expectation of a male life aged 10, is (by Table XLIV.) 43.94.

The *second* value (or the value deduced from this expectation by Prob. I.) is 17.838.

The *third* value (or the value deduced from the same expectation at 3 *per cent.*) is 21.277.

The difference between the first and second is .836; which (since the first is greater than the second) must be added to the *third;* and the *sum*, (that is, 22.113) will be the value required.

The *third* value at 5 *per cent.* is 15.286; and the difference added to 15.286 makes 16.122 the value of a male life aged 10 at 5 *per cent.* according to the SWEDEN observations.—The exact value at 5 *per cent.* is (by Table XLVII.) 16.014.

Again. The difference between 16.014 (the correct value at 5 *per cent.*) and 15.286 (the value at the same interest deduced from the expectation) is .728; which, *added* (because the first value is greater than the second) to 13.335, (the value deduced at 6 *per cent.* from the expectation) gives 14.063, the value of the same life, reckoning interest at 6 *per cent.*

These deductions, in the case of single lives particularly, are so easy, and give the true values so nearly, that it will be scarcely ever necessary to calculate the *exact* values (according to any given observations) for more than *one* rate of interest.

If, for instance, the correct values are computed at 4 *per cent.* according to any observations, the values at 3, $3\frac{1}{2}$, $4\frac{1}{2}$, 5, 6, 7, or 8 *per cent.* may be deduced from them by the preceding rules, as occasion may require, without much labour or any danger of considerable errors.—The following comparisons will shew in some measure how far these deductions may be depended on.

Values

Value of Single Male Lives by the SWEDEN Table of Observations.

Ages.	Values at 5 per cent. deduced from the correct values by Table XLVII. at 4 per cent.	Correct values by Table XLVII. at 5 *per cent.*
5	15.879	15.786
10	16.122	16.014
15	15.707	15.624
30	13.909	13.889
60	7.969	7.963
70	5.417	5.389

Values of two joint Lives by the NORTHAMPTON Table of Observations.

Ages.	Values at 5 *per cent.* deduced from the correct values at 4 *per cent.* by Table XX. &c.	Correct values at 5 *per cent.* by Tables XVIII. XIX. &c.	Values at 3 *per cent.* deduced from the correct values at 4 *per cent.* by Table XX. &c.	Correct values at 3 *per cent.* by Tables XX. XXI. &c.
5- 5	11.989	11.984	15.618	15.638
15-15	11.986	11.960	15.184	15.229
25-25	10.775	10.764	13.389	13.383
40-40	9.006	9.016	10.756	10.764
60-60	5.842	5.888	6.692	6.606
15-40	10.214	10.205	12.368	12.459
30-60	7.285	7.292	8.396	8.378
50-60	6.555	6.568	7.471	7.461

Values of Single Lives by the NORTHAMPTON Table of Observations.

Ages.	Values at 5 *per cent.* deduced from the correct values at 4 *per cent.* by Table XIX.	Correct values at 5 *per cent.* by Table XIX.	Values at 3 *per cent.* deduced from the correct values at 5 *per cent.* by Table XIX.	Correct values at 3 *per cent.* by Table XIX.
5	14.825	14.827	20.435	20.473
10	15.162	15.139	20.652	20.663
68	6.546	6.536	7.353	7.367

Ages.	Values at 4 *per cent.* deduced from the correct values at 5 *per cent.* by Table XIX.	Correct values at 4 *per cent.* by Table XIX.	Values at 5 *per cent.* deduced from the correct values at 3 *per cent.* by Table XIX.	Correct values at 5 *per cent.*
5	17.239	17.248	14.850	14.827
10	17.506	17.523	15.173	15.139
68	6.920	6.930	6.560	6.536

It may be observed in these examples, that the *deduced* values are *sometimes* almost the same with the *correct* values; that *generally* they do not differ more than a 20th or 30th of a year's purchase; that in *joint* lives they differ less than in *single* lives; and that they come equally near to one another *whatever* the rates of interest are.

The following observation will shew the reason of the circumstance last mentioned.

The value *deduced* from the expectation coincides with the *correct* value when the rate of interest is little or nothing; and, consequently, the difference between the two values becomes then little or nothing; and to this it is continually tending as the interest is diminished. On the contrary; the increase of value occasioned by the *de*crease of interest tends to make the difference greater. There is, therefore, in this case, the counter-action of two causes which always keep the difference nearly the same in all rates of interest.

The preceding rules seem to leave nothing wanting on this subject, except tables of the values of two joint lives *at any one rate of interest*, when the lives are either both male or both female lives. But the following rule for finding these values from the values in the SWEDEN Tables, will render the labour of composing such tables almost needless.

RULE for computing from the values of two joint lives in Tables XLVIII. and XLIX. the values of two joint lives both male or both female.

" Find in that column of Table XLV.
" which shews the expectations of *lives in*
" *general*, two ages whose expectations
" come nearest to the expectations of the
" two *male* or the two *female* lives pro-
" posed.

" From these expectations deduce, by
" the rule in p. 223, the value of two joint
" lives

" lives at those ages; and take the difference " between this value and the *correct* value at " those ages in Tables XLVIII. and XLIX.

" Deduce also, by the rule in page 223, " the value of the joint lives proposed, " from the expectations in Table XLV. of " male and female lives. The difference " just found *added* to this last value, if the " value before deduced from the expectations of *lives in general* is *less* than the " correct value, or *subtracted* from it if " *greater*, will be nearly the correct value of " the two joint lives proposed."

EXAMPLE.

Let the two proposed lives be both *female* lives, one aged 20 and the other aged 50.

The expectation of a *female* life aged 20 is, by Table XLIV. 39.15. The expectation nearest to it, in Table XLV. shewing the expectations of lives in general, is 39.47, corresponding to a life aged 18.——In like manner; the expectation in the same Table nearest to the expectation of a female life aged 50, is 19.09, corresponding to age 49.——The value (deduced from these expectations) of two joint lives aged 18 and 49, is, by the rule in p. 223, 10.245. The *correct* value, taken from Table XLIX. is 10.851, and the difference is .606, which difference *added* (since the former value is *less*

less than the latter) to 10.281 (the value of two joint female lives aged 50 and 20, deduced from the expectations by the rule in p. 223) makes 10.887, the *correct* value nearly of the joint *female* lives.

In order to find how near the values thus found come to the exact values, let the value of a *single female* life aged 20, (reckoning interest at 4 *per cent.*) be computed in the same manner from the correct values given in Table XLVII. of the values of lives in general.

The expectation in Table XLV. nearest to the expectation in Table XLIV. of a *female* life aged 20, is 39.47, which, in Table XLV. (shewing the expectation of lives in general) is the expectation of a life aged 18.———The value of a life aged 18, deduced from *this* expectation by the rule in p. 223, is 17.138. The correct value in that column of Table XLVII. which shews the values of lives in general, is 17.897. The former value is the least, and the difference is .759.———The value deduced by the same rule from 39.15 (the expectation of a *female* life aged 20) is 17.083, and the difference just found *added* to this value, makes it 17.842, which is very nearly the same with 17.872, the correct value in Table XLVII.——The value deduced in the same manner of a *male* life aged 20, is 17.363. The correct value (in Table XLVII.) is 17.335.

Value by *this Rule* of

A *female* life aged 50, is 12.000——Correct value is 12.049
aged 60, is 9.018——Correct value is 9.039
Of a *male* life aged 30, is 15.722——Correct value is 15.751
aged 70, is 5.702——Correct value is 5.670

In calculating by this rule, when any other rate of interest than 4 *per cent.* is used; the values of the joint lives, at that rate of interest, (deduced from the expectations and from the values in Tables XLVIII. and XLIX. at 4 *per cent.* by Prob. II. p. 223) must be taken for the *correct* values.——It must likewise be remembered, that this Rule cannot be used when the youngest of the two joint lives is *less* than ten years of age. In other cases, the values found by this Rule will be right *generally* within a 30th or 40th of a year's purchase, and *never*, I believe, wrong more than a 15th or 20th of a year's purchase.

CHAPTER V.

Short and easy Theorems for finding, in all Cases, the differences between the Values of Annuities payable Yearly, and of the same Annuities payable Half-yearly, Quarterly, or Momently. Communicated in a Letter to Sir John Pringle, *Bart. P.R.S. and read to the Royal Society, November 9, 1775, and published in the* Philosophical Transactions, *Vol. 66, Part. I.*

THE values of annuities, as given in all the common Tables, suppose them paid yearly. But it is well known, that generally they are paid half-yearly, and sometimes quarterly: and that this is a circumstance which always adds to their value. The difference between the values of annuities, according as they are paid in these different ways, I have seen no where stated with accuracy; and therefore, I have thought that the following attempt to do this may be of some use.

Annuities are of two sorts. They are either payable certainly or conditionally.

Of the former sort are all annuities which are payable at fixed times, without depending on any contingency. Of the latter sort are all annuities on lives. I will first consider the first sort of annuities.

Let r denote the interest of £1 for a year; and n the term or number of years during which any annuity is to be paid. Let P denote the value of the perpetuity, or the quotient arising from dividing £1 by its interest for a year. Let y denote the value of an an annuity for n years, supposing it to be paid yearly; h its value, payable half-yearly; q its value, payable quarterly; and m its value, payable momently.

THEOREM I.

$$y = \mathrm{P} - \frac{1}{r \times \overline{1+r}^{\,n}}$$

THEOREM II.

$$h = \mathrm{P} - \frac{1}{r \times \overline{1+\frac{r}{2}}^{\,2n}}$$

THEOREM.

THEOREM III.

$$q = \text{P} - \frac{1}{r \times \overline{1 + \frac{r}{4}}\Big|^{4n}}$$

THEOREM IV.

$\text{M} = \text{P} - \frac{r}{r\text{N}}$. where N denotes the number which hath rn for its hyperbolic logarithm, and $rn \times 0.43429448$ for its logarithm in *Brigg's* system.

EXAMPLE I.

Let the rate of interest be 4 *per cent.* and the term 5 years, and consequently $r = 0.04$, $n = 5$. $\text{P} = 25$.

Then, $y = 4.4518$
$h = 4.4913$
$q = 4.5120$
$m = 4.5415$

EXAMPLE II.

Let the rate of interest be the same, and the term for which the annuity is payable 25 years.

Then $y = 15.6220$
$h = 15.7118$
$q = 15.7694$
$m = 15.801$

EXAMPLE III.

Interest being the same, let the term be 50 years.

Then, $y = 21.4822$
$h = 21.5491$
$q = 21.582$
$m = 21.616$

EXAMPLE IV.

Interest being the same, let the term be 100 years.

Then, $y = 24.505$
$h = 24.523$
$q = 24.532$
$m = 24.542$

In the foregoing Theorems it may be observed, that the *ratio* to one another of the values of annuities payable yearly, half-yearly, quarterly, and momently, is greatest when n is least; that it decreases continually as n increases, till at last it vanishes when n becomes infinite or the annuity is a perpetuity. Agreeably to this it appears, in the examples I have given, that the values in the first example differ more from one another in proportion than the values in the second example; and that these also differ more than the values in the third; and that in the last example all the values are nearly the same.

These values computed by Mr. *De Moivre's* rules in his Treatise on Life-annuities, p. 86 and 124, &c. come out greater when *n* exceeds, and less when *n* falls short of 15 or 20 years. But those rules suppose the half-yearly and quarterly interests of money to be less than half or a quarter of the yearly interest. For instance; the value of an annuity of £1 payable half-yearly and quarterly for 50 years is, according to Mr. *De Moivre's* rules, 21.699 and 21.772, or a 99th part and 74th part more than the value of the same annuity payable yearly, supposing money improved at 4 *per cent.* when the annuity is paid yearly; and at £1.98 *per cent.* when it is paid half-yearly; and at £0.985 *per cent.* when it is paid quarterly: That is, supposing money improved at a rate of half-yearly or quarterly interest, which, instead of being a half or a quarter of the yearly interest, is only that half-yearly or quarterly payment which, in consequence of being laid up and improved at compound interest, will in a year amount to the sum that makes the yearly interest. It is obvious that this cannot be the proper method of computing these values. But not to insist on this; I will next state the different values of the second sort of annuities; or of *life-annuities*, according as they are supposed to be payable yearly, half-yearly, quarterly, or momently.

Let *r* as before be the interest of £1 for a year; *n* the complement of a given life;[z] *y*, *h*, *q*, and *m*, the values respectively of an annuity certain for *n* years payable yearly, half-yearly, quarterly, or momently; P the perpetuity; Y the present value of an annuity on a life whose complement is *n*, payable yearly; H the value of the same annuity payable half-yearly; and Q and M the values of the same annuity payable quarterly and momently.

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[z] The complement of a life is, in Mr. *De Moivre's* hypothesis, the number of years it wants of 86. In all other cases, it is double the expectation of a life; that is, it is double the quotient (diminished by $\frac{1}{2}$ unity) arising from dividing the sum of all the living in a Table of Observations from the age (inclusive) of the given life to the extremity of life, by the number of the living at that age.

Then, $$\text{Y} = \text{P} - \frac{1+r}{rn} \times y.$$

$$\text{H} = \text{P} - \frac{1+\frac{r}{2}}{nr} \times h.$$

$$\text{Q} = \text{P} - \frac{1+\frac{r}{4}}{nr} \times q.$$

$$\text{M} = \text{P} - \frac{m}{nr}.$$

EXAMPLE I.

Let the life be supposed of the age of 36. The complement of such a life is 50, according to Mr. *De Moivre's* hypothesis; and also very nearly, according to the *Breslaw* and the *Northampton* Tables of Observations. Therefore, n will be 50. Let the rate of interest be 4 *per cent.* or $r = 0.04$. $\text{P} = 25$. $y = 21.482$. $h = 21.549$. $q = 21.582$. $m = 21.616$. See p. 240.

Therefore, $$\text{Y} = 25 - \frac{1{,}04}{50 \times 0{,}04} \times 21{,}482 = 13{,}829$$

$$\text{H} = 25 - \frac{1{,}02}{50 \times 0{,}04} \times 21{,}549 = 14{,}010$$

$$\text{Q} = 25 - \frac{1{,}01}{50 \times 0{,}04} \times 21{,}582 = 14{,}101$$

$$\text{M} = 25 - \frac{21{,}616}{50 \times 0{,}04} = 14{,}191$$

EXAMPLE II.

Let the life be supposed of the age of 61. The complement of this life is 25 by Mr. *De Moivre's* hypothesis and the *Northampton* Table of Observations. Therefore, interest supposed at 4 *per cent.*

$$\text{Y} = 25 - \frac{1{,}04}{25 \times 0{,}04} \times 15{,}622 = 8{,}753$$

$$\text{H} = 25 - \frac{1{,}02}{25 \times 0{,}04} \times 15{,}712 = 8{,}973$$

$$\text{Q} = 25 - \frac{1{,}01}{25 \times 0{,}04} \times 15{,}769 = 9{,}072$$

$$\text{M} = 25 - \frac{15{,}801}{25 \times 0{,}04} = 9{,}199$$

The different values, given by these theorems[a] of life-annuities payable yearly, half-yearly, and quarterly, suppose nothing to be due to an annuitant for that year, half-year, or quarter, in which he shall happen to die. If, on the contrary, he is to be entitled to such part of the annuity as shall be proportioned to the time which shall happen to intervene between his death and the time when the payment immediately preceding his death became due; or in other words, if the annuity is an annuity secured by land, $\frac{y}{2n}$ must be added to the first theorem in order to obtain the value of such an annuity payable yearly. And in like manner, $\frac{h}{4n}$ must be added to the second theorem to obtain the value of the same annuity payable half-yearly: and $\frac{q}{8n}$ to the third theorem, to obtain its value payable quarterly.

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[a] It is of no consequence that these theorems are founded on the *hypothesis* of an equal decrement of life; for taking equal *yearly* values, (or values nearly equal) the differences between them and *half* yearly and *quarterly* values are almost exactly the same, whether they are deduced from real observations, or from this hypothesis.—— Even in the hypothesis itself it requires a considerable difference in the yearly value, to produce any material difference in the excess of the half-yearly and quarterly values.

The value, therefore, in the first example, of an annuity payable yearly on a life aged 36 being 13,829; its value, if secured by land, or to be enjoyed to the last moment of life, will be $13{,}829 + \frac{21{,}482}{100} = 14{,}043$. If secured by land and payable half-yearly, its value will be $14{,}010 + \frac{21{,}549}{200} = 14{,}117$. If secured by land and payable quarterly, its value will be $14{,}101 + \frac{21{,}582}{400} = 14{,}155$. The like values in the second example are 9,065, 9,130, and 9,151.

Life-annuities payable monthly or weekly may be considered as of the same value with annuities payable momently; and it is evident, that they must be enjoyed nearly to the last moment of life.

From these rules and examples it may be gathered, that the difference between the values of annuities on lives payable yearly, half-yearly, quarterly, and momently, increases continually with the ages; but, if not secured by land, this difference can never be so great as a quarter of a year's purchase in the case of annuities payable yearly and half yearly; three-eighths of a year's purchase in the case of annuities payable yearly and quarterly; and half a year's purchase in the case of annuities payable yearly and momently

Mr. *Simpson*, in his Treatise on the Doctrine of Life-annuities, p. 78, and in his Select Exercises, p. 283, hath given a quarter of a year's purchase as the addition always to be made to the value of a life-annuity payable yearly, in order to obtain its value payable half-yearly; and three-eighths of a year's purchase, if its value payable quarterly is required. But it appears, that these are too large additions; and, whatever be the rate of interest or the number of lives, a fifth of a year's purchase will be generally more than a sufficient addition, if the value of the annuity is desired payable half-yearly; and three-tenths of a year's purchase, if the value of the annuity is desired payable quarterly. Mr. *De Moivre's* rules, in p. 85 of his Book on Life-annuities, for finding the values of life-annuities payable half-yearly and quarterly from their values payable yearly, are still less correct; for they suppose the difference between these values the same, whether the annuities are life-annuities or annuities certain.

Mr. *Dodson*, in the first question in the third volume of his Mathematical Repository, hath given a rule for finding the value of an annuity secured by land and payable yearly, which coincides with that here given; and Mr. *De Moivre*, in p. 338 of his Treatise on the Doctrine of Chances, hath given a theorem for this purpose, which also brings out nearly the same answers. But Mr. *Simpson*, in Prob. I. p. 323 of his Select Exercises, makes the excess of the value of such an annuity above the value of an annuity payable yearly but not secured by land, double to the same excess derived from Mr. *Dodson's* and Mr. *De Moivre's* rules. The truth is, that Mr. *Dodson's* rule gives the exact value; and that Mr. *Simpson's* problem gives the value, not of an annuity secured by land and payable yearly, but of an annuity secured by land and payable momently; and also, that his method of solution implies a rate of interest somewhat less when the annuity is payable momently than when it is payable yearly.

But to prevent all perplexity on this subject, I will subjoin the following investigations, which will be easily understood by those who are acquainted with the common methods of calculating the values of life-annuities.

Let r, as before, be the interest of £1 for a year. Then the present value of £1 payable at the end of one year, two years, three years, &c. will be $\frac{1}{1+r}$, $\frac{1}{\overline{1+r}|^{2}}$, $\frac{1}{\overline{1+r}|^{3}}$, &c. respectively. And the present value of an annuity certain for n years payable yearly is the sum of this series continued to n terms,[b] or $\frac{1}{r} - \frac{1}{r \times \overline{1+r}|^{n}} = P - \frac{1}{r \times \overline{1+r}|^{n}} = y$.

In like manner, the present value of half £1 (that is, of 10*s.* = £0.5) payable at the end of half a year, a year, a year and a half, &c. reckoning half-yearly interest at half the annual interest, is $\frac{0,5}{1+\frac{r}{2}}$, $\frac{0,5}{\overline{1+\frac{r}{2}}\Big|^{2}}$, $\frac{0,5}{\overline{1+\frac{r}{2}}\Big|^{3}}$, &c.

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[b] In the Postscript it will be proved, that the sum of n terms of the series $\frac{1}{a} + \frac{1}{a^{2}} + \frac{1}{a^{3}} + \frac{1}{a^{4}}$, &c. is $\frac{1}{a-1} - \frac{1}{a^{n} \times \overline{a-1}}$. Substitute $1 + r$ for a, and it will appear that the sum of n terms of the series $\frac{1}{1+r} + \frac{1}{\overline{1+r}|^{2}} + \frac{1}{\overline{1+r}|^{3}}$, &c. is $\frac{1}{r} - \frac{1}{r \times \overline{1+r}|^{n}}$.

And the present value of an annuity certain payable half-yearly for n years, each payment to be half the yearly payment, is the sum of this series continued to $2n$ terms; or,

$$\frac{0,5}{\frac{r}{2}} - \frac{0,5}{\frac{r}{2}\times\overline{1+\frac{r}{2}}\Big|^{2n}} = \frac{1}{r} - \frac{1}{r\times\overline{1+\frac{r}{2}}\Big|^{2n}} = \text{P} - \frac{1}{r\times\overline{1+\frac{r}{2}}\Big|^{2n}} = h.$$

By the same steps it will appear, that the present value of an annuity certain for n years to be received in quarterly payments, each a quarter of the annual payment is,

$$\frac{0,25}{\frac{r}{4}} - \frac{0,25}{\frac{r}{4}\times\overline{1+\frac{r}{4}}\Big|^{4n}} = \text{P} - \frac{1}{r\times\overline{1+\frac{r}{4}}\Big|^{4n}} = q.$$

And also, that the present value of an annuity certain for n years, to be received in momently payments, each the same proportional part of the yearly payment that the moment is of the year, must be $\text{P} - \frac{1}{r\times\overline{1+\frac{r}{1000,\ \&c.}}\Big|^{1000,\ \&c.\ n}}$

But, by the binominal theorem,

$$\overline{1+\frac{r}{1000,\ \&c.}}\Big|^{1000,\ \&c.\ n} = 1 + rn + \frac{r^2n^2}{2} + \frac{r^3n^3}{2\times3} + \frac{r^4n^4}{2\times3\times4},$$

$\frac{r^4n^4}{2\times3\times4}$, &c. which series approximates indefinitely to the number of which rn is the hyperbolic logarithm, by Prob 1. Sect. XI. Vol. II. of Mr. *Simpson's* Fluxions; or by Prop. 1, p. 40, of his Treatise on Trigonometry. Therefore $P - \frac{1}{r \times \overline{1 + \frac{1}{1000,\ \&c.}}|^{1000,\ \&c.\ n}}$ $= P - \frac{1}{r.N} = m$, as expressed before. See p. 239.

If the value of an annuity of £1 for n years is required payable half-yearly, and the half-yearly interest of £1 instead of being half the yearly interest (or $\frac{r}{2}$), is supposed to be $\overline{1+r}|^{\frac{1}{2}} - 1$; the answer will be $\frac{0,5}{\overline{1+r}|^{\frac{1}{2}}} + \frac{0,5}{1+r} + \frac{0,5}{\overline{1+r}|^{\frac{3}{2}}} + \frac{0,5}{\overline{1+r}|^{2}}$, &c. continued to $2n$ terms $= \frac{0,5}{\overline{1+r}|^{\frac{1}{2}} - 1} - \frac{0,5}{\overline{1+r}|^{n} \times \overline{1 \times r}|^{\frac{1}{2}} - 1}$ $= \overline{1 - \frac{1}{\overline{1+r}|^{n}}} \times \frac{1}{2 \times \overline{1+r}|^{\frac{1}{2}} - 2}$; which value is to $\overline{1 - \frac{1}{\overline{1+r}|^{n}}} \times \frac{1}{r}$ (the value of the same annuity payable yearly supposing the yearly interest of £1 to be r) as $\frac{\frac{1}{2}}{\overline{1+r}|^{\frac{1}{2}}-1}$ to $\frac{1}{r}$,[b] agreeable to Mr. *De Moivre's* deduction in his Treatise on Life-annuities, p. 125, 4th edit.

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[b] In the same manner the value payable quarterly is $\overline{1-\frac{1}{\overline{1+r}|^{n}}} \times \frac{1}{4\times\overline{1+r}|^{\frac{1}{4}}-1}$ and the value payable momently $= \overline{1-\frac{1}{\overline{1+r}|^{n}}} \times \frac{1}{1000,\&c.\times\overline{1+r}|^{\frac{1}{1000},\&c.}-1}$. Consequently the value of an annuity certain, payable quarterly or momently, is to the same value, payable yearly, as $\frac{1}{4\times\overline{1+r}|^{\frac{1}{4}}-1}$, or $\frac{1}{N}$, to $\frac{1}{r}$ (N being the hyperbolic logarithm of $\overline{1+r}$). Supposing, therefore, the interest to be 4 *per cent.* the value of an annuity payable yearly must be *invariably* increased in the ratio of 1.0101, or 1.0152 or 1.01986 to 1, according as it is payable either half-yearly, quarterly, or momently. The difference, however, between the values of annuities payable yearly and at shorter intervals is known to be continually lessening in proportion to the length of the term, till at last, when the term is extended to a perpetuity, those values become the same, whether the payments are made yearly or momently. But such an equality can never take place according to Mr. *De Moivre's* rules; nay, if the term be extended only to 70 years, and interest be 6 *per cent.* an annuity payable quarterly will be worth more than even the *perpetuity* when the payments are made yearly. This is so manifestly wrong as to render any further argument unnecessary to expose the fallacy of Mr. *De Moivre's* method of solution. M.

This implying, in the case of annuities payable half-yearly, a smaller interest than half the yearly interest (for $\overline{1+r}|^{\frac{1}{2}} - 1$ is less than $\frac{r}{2}$) gives the difference between their value and the value of annuities payable yearly, greater than the truth.

But to return to the investigation of the theorems in the former part of this paper.

Let us again call P the perpetuity, and y the value of an annuity certain for r years and payable yearly; it is well known that the value of £1 payable yearly on a life whose complement is n is (supposing an equal decrement of life) $\frac{n-1}{n \times \overline{1+r}} + \frac{n-2}{r \times \overline{1+r}|^2} + \frac{n-3}{n \times \overline{1+r}|^3}$, &c. continued to n terms[c] $= \mathrm{P} - \frac{1+r}{nr} \times y = \mathrm{Y}$.

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[c] See Mr. *De Moivre's* Treatise on Life-annuities, p. 99, 4th edition. Or his Doctrine of Chances, p. 311, 4d edition. Or Mr. *Dodson's* Mathematical Repository, Vol. II. p. 137. Or Mr. *Simpson* on Annuities and Reversions, p. 14. In consulting these writers, care should be taken to remember, that they use r to denote the principal and interest of £1 for a year; whereas it hath been most convenient for me in these observations to make r stand only for the interest. In those writers, therefore, r signifies the same with $1 + r$ in this paper; and $r - 1$ the same with r.

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In like manner, supposing money improved at an half-yearly interest equal to half the yearly

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It is said above, that the value of an annuity payable yearly on a life whose complement is n, is $\frac{n-1}{n\times\overline{1+r}} + \frac{n-2}{n\times\overline{1+r}|^{2}} + \frac{n-3}{n\times\overline{1+r}|^{3}}$, &c. continued to n terms. This expression is equal to $\frac{n}{n\times\overline{1+r}} + \frac{n}{n\times\overline{1+r}|^{2}} + \frac{n}{n\times\overline{1+r}|^{3}}$, &c. $(n) - \frac{1}{n} \times \overline{\frac{1}{1+r} + \frac{2}{\overline{1+r}|^{2}} + \frac{3}{\overline{1+r}|^{3}}, \text{\&c.}}$ (n). But $\frac{n}{n\times\overline{1+r}} + \frac{n}{n\times\overline{1+r}|^{2}} + \frac{n}{n\times\overline{1+r}|^{3}}$, &c. $(= \frac{1}{1+r} + \frac{1}{\overline{1+r}|^{2}} + \frac{1}{\overline{1+r}|^{3}}$, &c.$) = \frac{1}{r} - \frac{1}{r\times\overline{1+r}|^{n}} = y$, see p. 248.) Also, by a theorem which will be demonstrated in the postscript, and putting a for any given quantity, $\frac{1}{a} + \frac{2}{a^{2}} + \frac{3}{a^{3}}$, &c. continued to n terms, $= \frac{a}{\overline{a-1}|^{2}} - \frac{n}{a^{n}} \times \frac{1}{a-1} - \frac{1}{a^{n}} \times \frac{a}{\overline{a-1}|^{2}}$. Therefore, if $1+r$ is substituted for a, and y for $\frac{1}{r} - \frac{1}{r\times\overline{1+r}|^{n}}$, the sum (multiplied by $\frac{1}{n}$) of n terms of the series $\frac{1}{1+r} + \frac{2}{\overline{1+r}|^{2}} + \frac{3}{\overline{1+r}|^{3}}$, &c. will come out $\frac{1+r}{nr} \times y - \frac{1}{r} \times \frac{1}{\overline{1+r}|^{n}}$; or $\frac{1+r}{nr} \times y + y - \frac{1}{r}$. Therefore, the series $\frac{1}{n} \times \frac{1}{1+r} + \frac{2}{\overline{1+r}|^{2}}$

yearly interest, or to $\frac{r}{2}$, the value of the same annuity payable half-yearly, is $\frac{1}{2}\times\overline{\frac{n-\frac{1}{2}}{n\times1+\frac{r}{2}}+\frac{n-1}{n\times\overline{1+\frac{r}{2}}|^{2}}+\frac{n-\frac{3}{2}}{n\times\overline{1+\frac{r}{2}}|^{3}}}$, &c. continued to $2n$ terms $=\frac{1}{2}\times\overline{\frac{n}{n\times1+\frac{r}{2}}+\frac{n}{n\times\overline{1+\frac{r}{2}}|^{2}}+\frac{n}{n\times\overline{1+\frac{r}{2}}|^{3}}}$, &c. continued to $2n$ terms $-\frac{1}{2}\times\overline{\frac{\frac{1}{2}}{n\times1+\frac{r}{2}}+\frac{1}{n\times\overline{1+\frac{r}{2}}|^{2}}+\frac{\frac{3}{2}}{n\times\overline{1+\frac{r}{2}}|^{3}}}$, &c. continued to $2n$ terms. But the sum of the first of these two series, or of $\frac{1}{2}\times\frac{n}{n\times1+\frac{r}{2}}+\frac{n}{n\times\overline{1+\frac{r}{2}}|^{2}}$, &c. (= $\frac{1}{2}$

$\overline{\frac{2}{\overline{1+r}|^{2}}+\frac{3}{\overline{1+r}|^{3}}}$, &c. continued to n terms and subtracted from the series $\frac{1}{1+r}+\frac{1}{\overline{1+r}|^{2}}+\frac{1}{\overline{1+r}|^{3}}$, &c. continued to n terms; that is, the value of the life will be $y-\frac{1+r}{nr}\times y+y-\frac{1}{r}=\frac{1}{r}-\frac{1+r}{nr}\times y=\mathrm{P}-\frac{1+r}{nr}\times y=\mathrm{Y}$.

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$\times\frac{1}{1+\frac{r}{2}}+\frac{1}{\overline{1+\frac{r}{2}}|^{2}}$, &c.) is h, see p. 249, &c.

And the sum of the second series is the same with half the sum of the series $\frac{1}{2n}\times\overline{\frac{1}{1+\frac{r}{2}}+\frac{2}{\overline{1+\frac{r}{2}}|^{2}}+\frac{3}{\overline{1+\frac{r}{2}}|^{3}}}$, &c. $(2n)$. But by the theorem mentioned in the last note, the sum of n terms of the series $\frac{1}{a}+\frac{2}{a^2}+\frac{3}{a^3}$, &c. is $\frac{a}{\overline{a-1}|^{2}}-\frac{n}{a^n}\times\frac{1}{a-1}-\frac{1}{a^n}\times\frac{a}{\overline{a-1}|^{2}}$. Therefore, if $1+\frac{r}{2}$ is substituted for a, $2n$ for n, and h for $\frac{1}{r}-\frac{1}{r\times\overline{1+\frac{r}{2}}|^{2n}}$, the sum of the second series (that is, of $\frac{1}{2}\times\frac{1}{2n}\times\overline{\frac{1}{1+\frac{r}{2}}+\frac{2}{\overline{1+\frac{r}{2}}|^{2}}+\frac{3}{\overline{1+\frac{r}{2}}|^{3}}}$, &c. $(2n)$ will come out $\frac{1+\frac{r}{2}}{nr}+h-\frac{1}{r}\times\frac{1}{\overline{1+\frac{r}{2}}|^{2n}}$, or $\frac{1+\frac{r}{2}}{nr}\times h+h-\frac{1}{r}$. Therefore, the second series subtracted from the first, leaves $\frac{1}{r} - \frac{1+\frac{r}{2}}{nr} \times h = \text{P} - \frac{1+\frac{r}{2}}{nr} \times h = \text{H}$, agreeably to the second theorem in p. 243.

By reasoning in the same way it may be easily found, that $\text{Q} = \text{P} - \frac{1+\frac{r}{4}}{nr} \times q$; and $\text{M} = \text{P} - \frac{1+\frac{r}{1000, \&c.}}{nr} \times m = \text{P} - \frac{m}{nr}$, agreeably to the third and fourth theorems in p. 243.

These theorems, I have said, suppose that an annuitant is entitled to no payment for that year, half-year, or quarter, in which he dies. If, on the contrary, he is to be entitled when he dies, to such a part of the yearly, half-yearly, or quarterly payment as shall bear the same proportion to the said payments respectively, as the intermediate time between the last payment and his death bears to the whole year, half-year, or quarter; in this case, supposing the annuity payable yearly, it is evident, since there is the same chance for his dying in one half of any year as in the other, that he will have an expectation of half a year's payment more than he would be otherwise entitled to. But the value of half £1 to be paid at the death of a person whose complement of life is n, is $\frac{1}{2} \times \frac{1}{n \times \overline{1+r}} + \frac{1}{2} \times \frac{1}{n \times \overline{1+r}|^2} + \frac{1}{2} \times \frac{1}{n \times \overline{1+r}|^3}$, &c. continued to n terms (d) $= \frac{y}{2n}$.

In like manner, a person who enjoys an annuity secured by land, payable half-yearly, will have an expectation of a quarter of a year's payment more than he could be otherwise entitled to; the value of which is $\frac{1}{4n} \times \frac{1}{1+\frac{r}{2}} + \frac{1}{\overline{1+\frac{r}{2}}|^2} + \frac{1}{\overline{1+\frac{r}{2}}|^3}$, &c. continued to $2n$ terms $= \frac{h}{4n}$. By the same reasoning it will appear, that $\frac{q}{8n}$ is the addition to be made to the value of an annuity payable quarterly, in order to obtain its value when secured by land.

IN the note, p. 248, the expression $\frac{1}{a-1} - \frac{1}{a^n} \times \frac{1}{a-1}$ is given as the sum of n terms of the series $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4}$, &c. to $\frac{1}{a^n}$; and the expression $\frac{a}{\overline{a-1}|^2} - \frac{n}{a^n} \times \frac{1}{a-1} - \frac{1}{a^n}$

(d) See page 248, &c.

$\times \frac{a}{\overline{a-1}|^{2}}$ is given, in p. 253, as the sum of n terms of the series $\frac{1}{a}+\frac{2}{a^2}+\frac{3}{a^3}+\frac{4}{a^4}$, &c.

The following investigation of these theorems being very easy, will not, perhaps, be unacceptable to those who have studied this subject.

Put $A = \frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}$, &c. $\frac{1}{a^n}$. $B = \frac{1}{a}+\frac{2}{a^2}+\frac{3}{a^3}+\frac{4}{a^4}$, &c. $\frac{n}{a^n}$.

Then $A \times a = 1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}$, &c. to $\frac{1}{a^{n-1}}$,

and $A \times a - 1 + \frac{1}{a^n} = \frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}$, &c. to $\frac{1}{a^{n-1}}+\frac{1}{a^n} = A$,

and $A \times a - A \; (= A \times \overline{a-1}) = 1 - \frac{1}{a^n}$.

Therefore, $A = \frac{1}{a-1} - \frac{1}{a^n} \times \frac{1}{a-1}$, which is the first theorem.

Again, $A \times a = 1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}$, &c. to $\frac{1}{a^{n-1}}$,

and $B \times a = 1+\frac{2}{a}+\frac{3}{a^2}+\frac{4}{a^3}$, &c. to $\frac{n}{a^{n-1}}$.

Therefore, $B \times a - A \times a = \frac{1}{a}+\frac{2}{a^2}+\frac{3}{a^3}$, &c. to $\frac{n-1}{a^{n-1}}$.

To both sides of the last equation add $\frac{n}{a^n}$, and it will appear, that

$$\text{B} \times a - \text{A} \times a + \frac{n}{a^n} = \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \frac{4}{a^4}, \text{ \&c. to } \frac{n-1}{a^n - 1} + \frac{n}{a^n} = \text{B}.$$

Therefore, $\text{B} \times a - \text{B} = \text{B} \times \overline{a-1} = \text{A} \times a - \frac{n}{a^n}$; and $\text{B} = \frac{\text{A} \times a}{a-1} - \frac{n}{a^{n+1} - a^n}$.

For A, in this last equation, substitute its equal, or $\frac{1}{a-1} - \frac{1}{a^n} \times \frac{1}{a-1}$, and the resulting equation will be $\frac{a}{a-1} - \frac{n}{a^n} \times \frac{1}{a-1} - \frac{1}{a^n} \times \frac{a}{\overline{a-1}|^2} = \text{B}$, which is the second theorem.

When n is infinite, all but the first terms in both these theorems vanish; and therefore, $\frac{1}{a-1}$ is the sum of the series $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, &c. continued infinitely; and $\frac{a}{\overline{a-1}|^2}$ is the sum of the series $\frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3}$, &c. continued infinitely.

By a like deduction, putting

$$\text{C} = \frac{1}{a} + \frac{2 \times 2}{a^2} + \frac{3 \times 3}{a^3} + \frac{4 \times 4}{a^4}, \text{ \&c. to } \frac{n^2}{a^n},$$

and D $= \frac{1}{a} + \frac{2\times2\times2}{a^2} + \frac{3\times3\times3}{a^3} + \frac{4\times4\times4}{a^4}$, &c. to $\frac{n^3}{a^n}$, it may be found that C $= \frac{A+2B+1}{a-1} - \frac{\overline{n+1}|^2}{a^{n+1}-a^n}$, and D $= \frac{A+3B+3C+1}{a-1} - \frac{\overline{n+1}|^3}{a^{n+1}-a^n}$. And consequently, substituting the values of A and B, that

$$C = \frac{a^2+a}{\overline{a-1}|^3} - \frac{n^2}{a^n} \times \frac{1}{a-1} - \frac{2an}{a^n} \times \frac{1}{\overline{a-1}|^2} - \frac{a^2+a}{a^n} \times \frac{1}{\overline{a-1}|^3}.$$

And substituting the values of A, B, C, that

$$D = \frac{a^3+4a^2+a}{\overline{a-1}|^4} - \frac{n^3}{a^n} \times \frac{1}{a-1} - \frac{3an^2}{a^n} \times \frac{1}{\overline{a-1}|^2} - \frac{3a^2n+3an}{a^n} \times \frac{1}{\overline{a-1}|^3} - \frac{a^3+4a^2+a}{a^n} \times \frac{1}{\overline{a-1}|^4}.$$

Or, since all but the first terms in these expressions vanish when n is infinite, that the sum of the series $\frac{1}{a} + \frac{4}{a^2} + \frac{9}{a^3}$, &c. continued infinitely is $\frac{a^2+a}{\overline{a-1}|^3}$; and that the sum of the series $\frac{1}{a} + \frac{8}{a^2} + \frac{27}{a^3} + \frac{64}{a^4}$, continued infinitely is $\frac{a^3 \times 4a^2 + a}{\overline{a-1}|^4}$.

These are all the theorems necessary for calculating the values of annuities on single lives, and on any two or three joint lives, upon the hypothesis of an equal decrement of life.

Supposing r the interest of £1 for a year, the sum of n terms of the series $\frac{1}{1+r} + \frac{1}{\overline{1+r}|^2} + \frac{1}{\overline{1+r}|^3}$, &c. is the present value of an annuity certain for n years; and $\frac{1}{1+r} + \frac{2}{\overline{1+r}|^2} + \frac{3}{\overline{1+r}|^3} + \frac{4}{\overline{1+r}|^4}$, (continued to n terms) is the present value of an annuity certain beginning with £1, and increasing to £2 the second year, to £3 the third year, &c.

If this last annuity is not an annuity certain for a given term, but a life-annuity, the value of it (supposing n the complement of the life, A the value of an annuity certain for n years, G the value of two equal joint lives whose common complement is n, P the perpetuity, and p the value of £1 to be received at the end of n years) will be $\overline{\text{A} - \text{G}} \times n + np\text{P} - \text{AP} \times \overline{1+r}$.

EXAMPLES.

Let the term be forty-one years, and the rate of interest 4 per cent.

The value of an annuity of £1 certain for this term is £20.

The value of an annuity certain for the same term, and beginning with £1 at the end of the first year, but increasing to £2 at the end of the second year, to £3 at the end of the third year, and so on till it becomes £41 at the end of the forty-first year, is (by the Second Theorem, putting 1 + r, or 1.04 for a) £314 ... 10*s.*

The value of an annuity increasing at this rate without end is £650.

If the annuity is a life-annuity which is to increase at the rate of £1 every year during the whole possible continuance of a life whose complement is forty-one years (or whose age, according to Table VI. in the next volume, is forty-five), the present value of it will be, by the last theorem, £135. But a much simpler rule for finding the values of annuities of this sort will be given in the following notes. See Note I; and also Mr. *Morgan* on Assurances, p. 119.

POSTSCRIPT.

THE first four Tables in the next volume furnish with the means of determining easily the values and amounts of sums and annuities for the most common *half*-yearly as well as *yearly* rates of interest; Mr. SMART, in his very useful and comprehensive Tables, having given the *half*-yearly values and amounts improperly.

It is very obvious, that the amount at any given *yearly* interest of any given annuity payable *half*-yearly, is the same with the amount of *half* that annuity at half the interest, and payable a *double* number of times. The amount, for instance, at 4 *per cent.* of an annuity of £10 payable *yearly* for 30 years, is, by Table IV. £560.849.

If it is payable *half*-yearly, its amount will be the same with the amount at 2 *per cent.* of an annuity of £5 payable for 60 years, which, by the same Table, is £570.257. ——In like manner; the amount at 5 *per cent.* of an annuity of £50 for 40 years, payable *half*-yearly, is the same with the amount at 2½ *per cent.* of an annuity of £25 for 80 years, which appears to be £6209.567. The amount at 5 *per cent.*

of the same annuity payable *yearly*, appears to be £6039.988.

Farther. The amount of £10 *principal* put out to *yearly* interest at 4 *per cent.* and forborne for 30 years, is (by Table III.) £32.433. But if it is put out to 4 *per cent. half*-yearly interest, its amount will be the same with the amount of the same principal, bearing *half* the interest in double the time; that is, it will, in the present instance, be the same with the amount of £10, bearing 2 *per cent.* interest in 60 years, which, by the Table last mentioned, appears to be £32.810.

These amounts can be thus determined from these Tables only, when the term for which they are wanted does not exceed 50 years, or 100 *half* years.

In order to find them for any longer term, the following method must be taken:

"If the amount required is the amount
"not of an *annuity*, but of a sum—find first
"the *half*-yearly amount for 50 years; after
"which find the *half*-yearly amount of that
"amount for the remainder of the term, and
"this last will be the amount desired."

EXAMPLE.

Let the amount be required, at 4 *per cent.* of £10 in 80 years, supposing the interest payable *half-yearly*.

Ans. The amount in 50 years, determined in the manner just described, is 72.446; and the amount of 72.446 in 30 years, determined in the same way, is 237.676, which is the amount required.

This amount, supposing the interest payable yearly, is £230.049.

But if the amount required is the amount of an *annuity* improved at any given rate of compound interest payable *half*-yearly, it will be necessary, after finding the sum which is the amount for 50 years, to find the yearly interest that sum will carry at the given rate; and the amount for the remainder of the term, of this interest increased by the annuity, added to the amount for 50 years, will be the amount required.

EXAMPLE.

Let the amount be required, at 4 *per cent.* of £10 *per ann.* in 80 years, supposing the annuity payable *half*-yearly.

Ans. The amount in 50 years (being the same with the amount of £5 *per ann.* in 100 years, at 2 *per cent.*) is, by Table IV. £1561.116.———The yearly interest of £1561.116, at 4 *per cent.* is £62.446, which increased by £10 makes £72.446; and the amount of £72.446 *per ann.* payable *half*-yearly in 30 years (or of £36.223 in 60 *half* years) is £4620.96, which added to £1561.116,

£1561.116, makes £6182.076 the amount required.

This amount, supposing the annuity payable yearly, is £5982.665.

N. B. These amounts for any given term and rate of interest are the same with the *debts* bearing that interest, which will be gradually sunk in that term by any given annuity appropriated to the redemption of the debt.—It appears, therefore, from the last example, that a sinking fund of a million *per ann.* never diverted would pay off, in 82 years, a public debt of 598 millions, bearing 4 *per cent.* interest, supposing it applied to that purpose *yearly*; but that if applied *half*-yearly, it would pay off, in the same time, a debt of 618 millions.

These examples shew the method of finding, by the abovementioned Tables, the values at any rate of interest of annuities payable for any given terms, supposing them payable *half*-yearly; and likewise the values of any sums payable at the end of any terms, supposing a *half*-yearly instead of a *yearly* discount allowed. But in such cases, these Tables will be of no use, if the terms exceed 50 years, or 100 *half*-years; and it will be necessary to have recourse to the *theorems* at the beginning of the preceding chapter, by which, with the help of logarithms, it is easy, in all cases, to compute the difference between the values of annuities

ties (including *life*-annuities) as they are payable yearly, half-yearly, or quarterly.

With respect to life-annuities, it may be proper to observe here particularly, that their values deduced from the *complements*, that is, from twice the *expectations* according to any given table of mortality (by the rules in p. 221, and p. 223), and payable *half*-yearly or *quarterly*, is the same with the values of *half* or a *quarter* the annuities at *half* or a *quarter* the yearly interest, deduced (by the same rules) from *double* or *quadruple* the complements: and that the difference between the *yearly* values and these *half*-yearly or *quarterly* values added to the *true* yearly values according to the same table of mortality, will give, with almost perfect correctness, the *half*-yearly or *quarterly* values according to that table.

EXAMPLE.

Let the *different* values be required of an annuity on a single life aged 50, according as it is payable *half*-yearly or quarterly, reckoning interest at 4 *per cent.* and the probabilities of the duration of life, as they are in Table XLV. Vol. II.

Ans. The *complement* (that is, twice the *expectation*) by Table XLV. of a life aged 50 is 36.92.——The value of an annuity payable

able *yearly* on a life at this age deduced at 4 *per cent.* from this *complement*, is by the rule in p. 221, 11.533.——The value deduced from *double* this complement (that is, of a life whose complement is supposed 73.84) at 2 *per cent.* is 23.466[c], the half of which is

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[*] In computing in this case, by the rule here referred to, it is necessary to find the value at 2 *per cent.* of an annuity certain payable for 73.84 years. This value (by the First Theorem in the preceding Chapter) is $50 - \frac{1}{.02 \times \overline{1.02}|^{73.84}}$. By logarithms it is easy to find that $\overline{1.02}|^{73.84}$ is 4.3154; and, consequently, that this expression is 38.416, which multiplied, according to the rule in p. 221, by 51 (the perpetuity increased by unity), and the product divided by 73.84 will give 36.533, which quotient subtracted from 50, (the perpetuity) leaves 23.466.

In like manner the value, by the same theorem, at 1 *per cent.* of an annuity certain for a number of years equal to four times the complement (that is, to 147.68 years) is $100 - \frac{1}{.01 \times \overline{1.01}|^{147.68}}$, which is equal to 76.994; and the product of 76.994 into 101, divided by 147.68, gives 52.654, which, subtracted from 100, leaves 47.345, the quarter of which is 11.836.

It is necessary to add here, that in computing the yearly value of any life-annuity from the expectation by the rule in p. 221, the value of an annuity certain for a number of years equal to twice the expectation (or the complement) may be always taken from Table II. Vol. II. when the complement is any *whole* number of years; and also, that when it is not any whole number of years, it may be taken for the correspondent arithmetical mean between the two nearest yearly values in the Table. Thus; in the example given above, the value at 4 *per cent.* of an annuity certain for 36 years, by Table II.

is 11.733.——The difference is .200. And this difference, added to 11.658 (the *true* value by Table XLVII. of an annuity payable yearly on the supposed life), makes 11.858; which is the true value of the annuity payable *half*-yearly.

The value of the same annuity deduced (by the rule in p. 221) from *quadruple* the complement at 1 *per cent.*; that is, the value at 1 *per cent.* of a life whose complement is supposed to be 147.68 years, is 47.345, the quarter of which is 11.836. The difference between this value and 11.733 is .303, which added as before to 11.658, makes 11.961, the true value of the annuity payable quarterly.

In the same way the values are to be computed (by the second rule in p. 223) of annuities payable *half-yearly* or *quarterly* on any two joint lives.

If the annuity is a life-annuity[f] secured on

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is 18.908. The value for 37 years is 19.142. The difference is .234; and this difference multiplied by .92 (the fractional part of the complement) and added to the least of these two values, gives 19.123 for the value of an annuity certain for 36.92 years.

The exact value by the first Theorem is 25 − $\frac{1}{.04 \times \overline{1.04}|^{36.92}}$, which is equal to 19.123, and the same with the former value.

[f] It should be remembered, that all the values of life-annuities in the Tables in the next volume suppose that, when

on land, the value is to be computed by the directions in the preceding chapter.——— If such an annuity is payable *half*-yearly, as is most common, its value, in the present instance, will be 11.858 (the half-yearly value just determined) increased by the quotient of 19.206 (the value of an annuity certain and payable *half*-yearly, for a number of *half* years, equal to *four* times the *expectation* or *twice* the *complement*; that is, 73.84) divided by *four* times the complement. This quotient is .130; and the value, therefore, is 11.989.

The following comparison will shew, in some measure, what additions should be made, at all ages, to the yearly values of life-annuities, on account of these different modes of payment.

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when the annuitant dies, nothing can be claimed for the time that has past since the last payment became due. If a payment proportioned to that time may be claimed; that is, if the annuity is payable to the last moment of life, it is called an annuity *secured on land*.

TABLE

Shewing the Additions to the Values of Life-Annuities on account of their being payable *half-yearly*, or *quarterly*, or *half-yearly* and secured on Land.

Interest 4 *per cent.*

Age	Yearly value by Table XLVII.	Half-yearly value.	Excess above yearly value.	Quarterly value.	Excess above yearly value.	Value secured by land and payable half-yearly.	Excess above yearly value.
10	18.891	19.018	.127	19.089	.198	19.085	.194
20	17.603	17.746	.143	17.819	.216	17.824	.221
30	16.006	16.168	.162	16.249	.243	16.259	.253
40	14.034	14.216	.182	14.307	.273	14.324	.290
50	11.658	11.858	.200	11.961	.303	11.989	.331
60	8.789	9.014	.225	9.119	.330	9.170	.381
70	5.783	6.019	.236	6.136	.353	6.204	.421
75	4.534	4.770	.236	4.892	.358	4.967	.433

Interest 5 *per cent.*

Age							
10	16.169	16.278	.109	16.331	.162	16.382	.163
20	15.260	15.383	.123	15.445	.183	15.447	.187
30	14.080	14.223	.143	14.293	.213	14.299	.219
40	12.558	12.720	.162	12.801	.243	12.812	.254
50	10.634	10.820	.186	10.914	.280	10.933	.299
60	8.184	8.403	.219	8.498	.314	8.544	.360
70	5.494	5.719	.225	5.835	.341	5.891	.397
75	4.345	4.575	.230	4.692	.347	4.762	.417

These excesses are the same from whatever tables of mortality the yearly values are deduced.——They are also nearly the same (supposing equal *yearly* values) whether the yearly values are the values of single, or of joint-lives, or of any number of lives.

CHAPTER VI.

Of PUBLIC CREDIT, *and the* NATIONAL DEBT.

THE *National Debt* is a subject in which the public is deeply interested. Some observations have occurred to me upon it, which I think important; and for this reason, though foreign to my chief purpose in this work, I shall beg leave to offer them to public attention.

The practice of raising the necessary supplies for every national service, by borrowing money on interest, to be continued till the principal is discharged, must be in the highest degree detrimental to a kingdom, unless a plan is settled for putting its debts into a regular and certain course of payment. When this is not done, a kingdom, by such a practice, obliges itself to return for every sum it borrows infinitely greater sums; and, for the sake of a present advantage, subjects itself to a burden which must be always growing heavier and heavier, till it becomes insupportable.

This seems to be now the very state of

this nation. At the REVOLUTION, an æra in other respects truly glorious, the practice I have mentioned begun. Ever since, the public debt has been increasing fast, and every new war has added much more to it than was taken from it during the preceding period of peace. In the year 1700, it was 16 millions. In 1715, it was 55 millions. A peace, which continued till 1740, sunk it to 46 millions; but the succeeding war increased it to 78 millions; and the next peace sunk it no lower than 75 millions. In the *last* war it rose to 146 millions and a half. During a peace which has lasted now 10 years, it has been reduced to 138 millions: and at a sum not much less than this, it will, perhaps, be found at the commencement of another war, which may possibly raise it to 200 millions[g].—One cannot reflect on this without terror.—No resources can be sufficient to support a kingdom long in such a course. 'Tis obvious, that the consequence of accumulating debts so rapidly, and of mortgaging posterity, and funding for eternity, in order to pay the interest of them, must, in the end, prove destructive. Rather than go on in this way, it is absolutely necessary, that no money should be borrowed, except on annuities, which are to terminate within a given period. Were this practised, there would be a LIMIT beyond which the national debts could not increase; and time would do that necessarily for the public, which, if trusted to the economy of the conductors of its affairs might possibly *never* be done.

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[g] It should be remembered that this was written in 1773.—In 1774 and 1775, two millions of the capital of the 3 *per cent.* annuities were paid off, which reduced the capital of the public debt to 136 millions; and at this sum, nearly, (supposing the long annuity then worth 27 years purchase) it stood at the beginning of the *American* war.(*)

(*) The additions which have been made to the public debt by the American, and the late war, together with its whole amount at the *beginning* of the present war in 1802, may be learnt from the account at the end of this Chapter. What the amount may be at the *conclusion* of this war it is neither easy nor agreeable to conjecture. M.

This, therefore, is one of the proposals to which on this occasion, I wish I could engage attention. I am sensible, indeed, that the *present* burdens of the state would, in this case, be increased, in consequence of the greater present interest which would be necessary to be given for money: but I do not consider this as an objection of any weight. For let the annuity be an annuity for 100 years. Such an annuity is, to the present views of men, nearly the same with an annuity for ever; and it is also nearly the same in calculation, its value at 4 *per cent.* being 24½ years purchase, and therefore only half a year's purchase less than the value of a *perpetuity*. Supposing, therefore, the public able to borrow money at 4 *per cent.* on annuities for ever, it ought not to give above 1*s.* 7*d. per cent.* more for money borrowed on annuities for 100 years: But should it be obliged to give a *quarter*, or even a *half per cent.* more[h], the additional burdens derived from hence, would not be such as could be very sensibly felt; and the advantages arising from the necessary annihilation of the public debts by time, would abundantly overbalance them.

These advantages would be, indeed, unspeakably great. By such a method of raising money, the expence of one war would, in time, come to be always discharged before a new war commenced; and it would be impossible that a state should ever have upon it, at any one time, the expence of many wars; or any larger debts than could be contracted within the limited period of the annuities; and consequently, it would enjoy the invaluable privilege of being rendered, in some degree, independent of the management of its finances by ignorant or unfaithful servants.

I must add, that it is by no means necessary, that the limited period of the annuities should be so long as I have mentioned, or

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[h] The annuities might be kept 18 years without being much diminished in value; for, supposing interest at 4 *per cent.* an annuity for 82 years, is, within a 49th part, or £2 in £98 worth as much as an annuity for 100 years.

Perhaps, in this way of raising money, it might be best to offer a higher interest at first, which should fall to a lower, at the end of given intervals. Thus, though 4½ for 100 years is equal in value to 5 *per cent.* for 17 years, and after that 4 *per cent.* for 83 years, yet the latter might appear more inviting.

100 years; and that, at any time before the expiration of this period, the public might employ any surplus monies, in extinguishing part of the annuities, by purchasing them for itself at the market price; and thus it might aid the operations of time, and keep its debts within any bounds that its interest rendered necessary. Our government has, I know, in some instances adopted the plan now proposed; but it is to be wished that, instead of retracting[1] it, as was once done, it had been carried much further.

I am, however, far from intending to recommend this plan as the best a state can pursue. There is another method of gaining the same end, which is, on many accounts preferable to it. I mean, "by providing an "annual saving, to be applied invariably, "together with the interest of all the sums "redeemed by it, to the purpose of discharging the public debts: Or, in other "words, by the establishment of a permanent "SINKING FUND."

It is well known, that this plan has been also adopted by our government; but, though capable of producing the *greatest* effects in the *easiest* and *surest* manner, it has never been carried into execution. It will abundantly appear, from what follows, that this observation is just.

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[1] In the year 1720, the nation was put to the expence of above three millions, in order to reduce several long and short annuities then subsisting to redeemable *perpetuities*.

Suppose the annual saving to be £100,000. This sum, applied *now* to discharge an equal debt, bearing interest at 4 *per cent.* will transfer to the public, from its creditors, an annuity of £4000.* The annual saving, therefore, would be increased to £104,000; and this saving would transfer to the public another annuity of £4,160, and make the saving at the beginning of the 2d year, to be £108,160.—Thus, the original fund would go on increasing, at the same rate with money improved at 4 *per cent.* compound interest.—At the beginning of the 3d year it would be £112,486. At the beginning of the 18th year, £202,581. Of the 36th year, £410,393, and of the 95th year, £4,151,138.—In 94 years, then, the nation might be eased of above 4 millions *per annum* in taxes; and above 100 millions of its debts would be discharged, gradually and insensibly, at no greater expence than £100,000 *per annum*; and without interfering with any of the resources of government; or making any other difference, than causing *funds* to be engaged for a course of time to the *public*, which would have been otherwise necessarily engaged to its *creditors*, and which, therefore, must have been entirely useless to it.

It is an observation that deserves particular attention here, that on this plan, it will be of less importance to a state what interest it is obliged to give for money: For the higher the interest, the sooner will such a fund pay off the principal. Thus; a hundred millions borrowed at 8 *per cent.* and bearing an annual interest of eight millions, would be paid off by a fund, producing annually £100,000 in 56 years; that is, in 58 years less time than if the same money had been borrowed at 4 *per cent.*[k]

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[k] What is here said, supposes the *same* fund applied to the discharge of debts bearing *different* interests. If different funds are applied, bearing to one another the same proportion with the interests of the debts which they are to disharge, the benefit derived from borrowing on lower rather than higher interests, will be reduced to almost nothing; for the disbursements of the public on account of all equal loans, will, in this case, be nearly the same.

The following example will explain and demonstrate this:

Let a million be borrowed at 3 *per cent.* and let a fund be charged with it, bringing in *six shillings per cent. per ann.* more than the interest; or £33,000 instead of £30,000 *per ann.* This *surplus, unalienably* applied, together with all the interests disengaged by it, will annihilate the *principal* in 81 years, as may be gathered from Table IV. in the next volume. And the disbursements, on account of the loan, will be 81 multiplied by £33,000; that is, £2,673,000. Let us suppose again, a million borrowed at 6 *per cent.* and let a fund be charged with it, producing a surplus of *twelve shillings per cent. per ann.*; such a fund, besides paying the interest, will discharge the *principal* in 41 years; and the disbursements, on account of the loan, will be £66,000 multiplied by 41; that is, £2,706,000, or nearly the same with the disbursements on account of an equal loan at 3 *per cent.*

It follows from hence, that reductions of interest would on this plan, be no great advantage to a state. They would, indeed, lighten its present burdens; but this advantage would be, in some measure, balanced by the addition which would be made to its *future* burdens, in consequence of the longer time during which it would be necessary to bear them.—I mean this on the supposition, that the savings produced by reductions of interest, are immediately applied to the relief of the state, by annihilating taxes equivalent to them. But if that is not the case; and if, likewise, there is either no plan established for putting the public debts into a certain course of payment, or it is not faithfully carried into execution; in these circumstances, reductions of interest may prove hurtful. For, first, They would only furnish funds for contracting further debts, and with more money for supplying the deficiencies arising from profusion and bad management. And, secondly, As, in such circumstances, they would only *retard*, and not *prevent* the increase of the burdens occasioned by the public debts, a period would come when the affairs of the state would get near to a *crisis*; and at such a period, its danger would be increased, in proportion to the reductions of interest that had been made.

In order to understand this; let us suppose that a debt, bearing an annual interest of five millions, is the whole debt which a state can bear without being so much oppressed as to be near sinking. Let it, however, be supposed to have still some last resources left, which may enable it to bear, for 23 years to come, this load, together with every additional load, which, during this time, may be necessary to be thrown upon it.—Let it further be supposed, that at this time, the state urged by the fear of an approaching bankruptcy, resolves upon entering into some effectual measure for preserving itself.—Certain it is, that in such circumstances, no measure *so* effectual can be pursued, as the establishment of a *sinking fund*, and such a faithful application of it as I have explained. Let that then be the measure entered upon; and let the state be supposed capable of providing a fund, producing a million annually. If all the debts bear interest at 6 *per cent.* this fund would pay off three-fifths of them, within the time I have mentioned; or, in 23 years; and the state might be saved. But if, in consequence of reductions, they bear interest at no more than 3 *per cent.* the same fund would not give the same relief, in less than *double* that time; and therefore, a bankruptcy might prove unavoidable[1].

I wish I could think, that there is nothing in this representation, that can be applied to the present state of this nation. The interest of the public debts has been reduced, at different periods, from 6 to 5, from 5 to 4, and from 4 to 3 *per cent.*; but still they have grown with rapidity; and we now see ourselves overloaded. and in no way of gaining relief Had there been no reductions of interest, we should, indeed, have been in the same condition sooner; but, we might have been relieved also sooner, and with less difficulty and danger.

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[1] It should not be forgotten, that the operation of such a scheme would raise the 3 *per cent.* debts nearly to *par*, and render the discharge of them at a discount impracticable.

In short. Reductions of interest are advantageous chiefly when made to gain additions to such a *sinking fund* as I have described.—When made with other views, they are only *palliatives*, which give *present* relief by increasing *future* danger; or *expedients* which *postpone* a public bankruptcy, by rendering it a calamity more *unavoidable* and *dreadful.* As managed therefore among us, they have been indeed the effects of too narrow a policy, and deserve none of the *encomiums* which have been bestowed upon them. The preceding observations prove this sufficiently; but there is one farther proof of it which I cannot help mentioning. Suppose £200,000 *per annum* to have been gained in 1716, by the reduction which was then made of the 6 *per cents.* to 5 *per cents.*; or, in other words, by saving 1 *per cent. per annum* on a capital of 20 millions. This saving, in consequence of being applied unalienably in the manner I have represented, to the payment of the public debts, would, in 37 years, have discharged a debt of £20,325,000, bearing 5 *per cent.* interest.

But if applied every year to current services, in order to avoid levying new taxes, the benefit derived from it in the same period would be 37 times £200,000, or £7,400,000; but at the same time, a debt would have been continued of 20 millions, which must have been otherwise paid. The effect, therefore, in this case, of the reduction, would be to prevent an incumbrance on the public of £200,000 *per ann.* by leaving upon it an incumbrance of a million *per ann.* rendered more difficult and unlikely than ever to be removed[m]

But

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[m] There is an inconvenience arising from reductions of interest, which I did not attend to when this Chapter was composed, but which is greater than any here mentioned. They have given rise to the extravagant *douceurs* or *premiums* which have been annexed to our public loans, by causing the lenders of money to consider whatever interest could be offered above 3 *per cent.* as no more than a short annuity.—Had there been no reduction of interest before the war which begun in 1755, the price of the *four per cent.* stocks would have been during the whole war either *above* par or not much *below* it, and government might have borrowed at this or a lower interest without *premiums*. But the reduction which had just taken place of 56 millions to an interest of 3 *per cent.* made it unavoidable for monied men to consider this stock as only a 3 *per cent.* stock with £1 *per ann.* added till peace was established; and this made it necessary for government, either to promise interest extravagantly high, or to offer long, or short, or life-annuities, or additional capitals, as rewards for lending money at 3 *per cent.* The last method has constantly been adopted; and in consequence of it, the public has been put to an enormous expence, which the avoiding of reductions and a little management would have rendered entirely unnecessary.

I have endeavoured to explain this in the second Tract on

But to return to the subject I have principally in view;

What I have said implies, that a state always discharges its debts, whatever interest they bear, by paying the original sum borrowed. It may, perhaps, be imagined, that when a loan is under *par*, it may be discharged at a less expence. But this is by no means so practicable as it may seem; for it should be considered, that a public loan, now under *par*, would not long keep so, after being put into a course of payments: And for this reason, as a state can never be obliged, in redeeming its debts, to pay *more* than the original sum borrowed, so neither ought it to expect,

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on Civil Liberty. Part II. Sect. III. and particularly in the Supplement to that Tract, where regulations are proposed for raising the value of stocks bearing higher interest than 3 *per cent.* in order to avoid the necessity of offering high *premiums* and creating artificial debts.—The following example will shew what occasion there is for some such regulations.——The 3 *per cent.* stocks were (Oct. 25th, 1781,) at 55¼; and the price of the 4 *per cent.* stocks, which, in order to bear a proportionable price, ought to have been 73¾, was (deducting a fortnight's interest) 70¼. One *per cent.* added to make a *five per cent.* stock, would have been still more undervalued; and such a stock, even with the advantage of being declared irredeemable for 15 or 20 years, would not probably have sold for more than £82, or £83. Had, therefore government attempted to borrow by offering 5 *per cent.* it could not have succeeded without giving a *premium* of near £20 for every £100 advanced. But, by such regulations as those proposed in the *Supplement* to which I have just referred, it would, I imagine, have become practicable to borrow at interests corresponding nearly to the prices of the 3 *per cents.* without *premiums.*

expect, in general, to be able to redeem them by paying *less*. I have said, *in general*; for I am sensible that, at the beginning of the operations of a fund, when its produce is small, and also in a time of war, a state might derive great advantages from the low price of its debts. And I am sensible also, that considerable advantages might be derived from *lotteries*, in paying the public debts: But *lotteries* do great mischief in a state, by fostering the destructive spirit of gaming. It is wretched policy to make them familiar, by recurring to them in the ordinary course of government. There are great occasions on which they may be necessary, and for such occasions they should be reserved.

The advantages of putting the public debts into such a course of payment as I have described, are scarcely to be imagined. It would give a vigour to public credit, which would enable a state always to borrow money easily, and on the best terms. And the encouragement to lenders might be always improved, without any inconvenience; by making every loan irredeemable, during the first 20 or 30 years; for, there could seldom be any occasion for beginning to discharge any *one* loan sooner.

It might be easily shewn; that the faithful application, from the beginning of the year 1700, of only £200,000 annually, would long before 1790, notwithstanding the reductions of interest, have paid off above a hundred millions of the public debts. The nation might, therefore, some years ago, have been eased of a great part of the taxes with which it is loaded. The most important relief might have been given to its trade and manufactures; and it might now have been in better circumstances than at the beginning of the last war; its credit firm; respected by foreign nations, and dreaded by its enemies. A new account might also some years ago have been begun; and another fund, not much more considerable, applied in the same way, would, in a short course of years, have paid all that would have been now unpaid of our debts. And thus, without any expence that could be sensibly felt, its debts, as soon as they began to grow heavy, might have been constantly reduced to a *half*, or a *third*; and not only all *danger*, but all considerable *inconvenience* from them prevented.

All I have now said, supposes a *single* fund with a *general* appropriation to the payment of the public debts. The same ends might be answered by *particular* funds, with small surpluses, appropriated to *particular* debts. In the wars of King *William* and Queen *Anne*, an interest of six, and sometimes seven and eight *per cent.* was given for loans. It would have been easy to have annexed to each loan a *fund* producing a *surplus* of £1 *per cent.* after paying the interest; and such a *surplus* would have been sufficient to annihilate the principal of every loan in 33 years. Had this plan been followed, the disengagement of the public funds, and the relief attending it would have begun 60 years ago; and the debts contracted, during the reigns of King *William* and Queen *Anne*, would have been all cancelled near 30 years ago, without any of that trouble, tumult, and distress, which have been occasioned by reductions of interest, and by the various schemes which have been tried for lessening the debts [n].—A fund, yielding £1 *per cent.* surplus, annexed to a loan of 5 *per cent.* would discharge the principal in 37 years [o]. At 4 *per cent.* in 41 years. At 3 *per cent.* in 47 years.

These observations relate only to what *might* have been the state of the nation with respect to its debts, had a right plan been pursued from the first. But it will be asked, What can be done with them as they *are*? I wish I was able to give a more satisfactory answer to this enquiry. Every one must see our prospect to be discouraging, and our state hazardous. Some have thought, that a good method might be found out of discharging the national debt, by short annuities, and life-annuities. The following observations will shew how groundless an imagination this is.

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[n] The sums to be laid out would, in this case, be so small at first, that it would be proper to employ them in purchasing part of the loan to be annihilated at the prices in the public market; and this, as far as it can be carried, is the most easy and quiet and silent way possible of extinguishing the public debts.

[o] I have all along supposed the produce of the public funds to come in yearly. The truth is, that it comes in *half*-yearly; but this gives no advantage in the payment of the public debts worth taking into account. £1 *per annum*, together with its growing interests, at 4 *per cent.* taken *yearly* out of £100, will reduce it to nothing in 41 years; if taken *half*-yearly, it will annihilate the same capital only four months and 12 days sooner.

Short annuities and life-annuities have been always undervalued by the public; and were they offered to sale to such an amount as would be necessary to make any considerable reduction in the national debt, they would probably fall to a very low price. Let the three *per cent.* stocks be supposed at 86. £100 of this stock would not be given up for a life-annuity less than £6, or a short annuity for a less term than 30 years; for this is valuing them at 14⅓ years purchase, and life-annuities in particular have never yet been disposed of at so high a price.—In order, therefore, to discharge in this way a capital in the 3 *per cent.* annuities of 33 millions and a third (the interest of which is a million *per ann.*) it would be necessary to provide a surplus producing another million *per ann.* for 30 years. But this surplus, employed during the same time as a sinking fund, would redeem 47 millions and a half at *par*, and 51 millions and a half at £86 in *money* for every £100 *stock*. It would, therefore, be great folly to employ such a surplus in the former way rather than the latter.

But I will beg leave to detain the reader here a little longer——The observations now made may be of use in shewing what the best method is of incurring debts as well as redeeming them.

Suppose a million raised by *annuities* on a set of lives, all at 30 years of age. The purchasers of such annuities cannot reasonably be reckoned to have an expectation of less than 30 years. That is; the duration of their lives, taking them one with another, will be 30 years; and they will be entitled, supposing interest at 4 *per cent.* to £7 *per ann.* for every £100 advanced. For a million then, the public would make 30 payments of £70 000.—Let us suppose next, that a fund producing this sum annually, instead of being engaged to pay these life-annuities, is engaged for 30 years, to pay the principal and interest of a million, borrowed on *redeemable* perpetuities, at 4 *per cent.* There will, at the end of the first year, be a surplus of £30,000.—In consequence of applying this to the extinction of the principal, it will be reduced to £970,000, on which, at the end of the second year, the interest due will be £38 800. There will, therefore, be a saving of £1200. Instead of employing this saving in further sinking the *principal*, which would cause the fund to accumulate in the same manner with money at compound interest, let it be taken and employed in any other way: And let the same be done with all the subsequent savings, reserving only £30,000 annually, for the purpose of sinking the principal. At the end of the second year, the principal will be £940,000;

and the saving of interest upon it, at the end of the third year, £2400. At the end of the 30th year, the principal will be reduced to £100,000. The saving of interest that year will be, £1200 multiplied by 29, or £34,800, and the sum of all the savings will be £522,000. Deduct from hence £100,000 remaining then undischarged of the principal; and £422,000 will be the loss the public would sustain in the circumstances I have supposed, by raising money ~~on life~~-annuities. But if we suppose the savings, as they arise, as well as the constant sum of £30,000, to be applied to the discharge of the principal, instead of being spent on current services; the whole million will be annihilated in 21 years and a half; and the loss to the public by preferring life-annuities, will be 8½ years purchase of the annuities; or £595,000. By similar deductions it may be easily found, that the loss, in *younger* lives, is greater; in *older* lives less; but never inconsiderable, except in the *oldest* lives.

It appears, therefore, that in consequence of such a way of raising money, the public must always pay much more in interest than there is any occasion for; and *waste* a sum equal to more than half the principal borrowed[p]. This, however, though so wasteful,

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[p] It is obvious, that the observations here made, may be applied to the common methods of raising money on life-annuities, for building churches, paving streets, making navigations, &c. &c. And, in general, to all cases

is a more frugal way of procuring money than by borrowing on *perpetuities*, without putting them into a course of redemption; for in this case the loss may be *infinite*.

But to return.

The enquiry which has occasioned this digression, must be interesting to every person who wishes well to this country. All

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where the money received is not laid up to be improved. —For, to view this subject in another light, let us suppose £10,000 borrowed for any public work, on perpetuities at 4 *per cent.* And, if that will afford more encouragement, let them be made irredeemable for any number of years less than seventeen. Let us further suppose such rates, or tolls, established for the payment of the interest and principal, as shall produce *double* the interest of the sum borrowed; or £800 *per annum* instead of £400 *per annum*. Let the *surplus*, as it comes in *half-yearly*, be laid up to accumulate in the public funds. In 17 years and a half, reckoning interest at 4 *per cent.* a capital will be raised, equal to the whole sum borrowed; and, therefore, at the end of that time, the whole debt may be discharged, and the whole transaction finished.—But if the same sum had been borrowed on annuities, for the lives of a set of persons 50 years of age, at 8 *per cent.* which is £1 *per cent.* less than the true value of such annuities: Had this, I say, been done, *half* the annuitants would have been alive at the end of the term I have mentioned; and the whole transaction, together with the expences and trouble attending the management of it, could not have been finally closed till the extinction of all the lives; that is, not in less time, most probably, than 35, or, perhaps, 40 years.—It is a necessary observation here, that, if public credit maintains its ground, much will not depend, in the plan now proposed, on the rise and fall of STOCKS. If a *war* sinks them, the money laid out, while the war lasts, will accumulate faster. If a peace raises them, the money that had been previously laid out will be proportionably increased.

schemes for discharging the public debts by life-annuities, have been shewn to be absurd and extravagant. In general it may be observed, that it is far from probable, that any money which the nation can spare, if applied so as to bear only *simple* interest, can be capable of reducing its debts within due bounds; or of doing us, in our present circumstances, any essential service. A fund, producing a surplus of even two millions annually, would, when thus applied, pay no more than 40 millions in 20 years; and, in that time, a war might probably come, which would interrupt the application of it; and increase our debts much more than such a fund had lessened them.

Certain it is, therefore, that if our affairs are to be retrieved, it must be by a *fund* increasing itself in the manner I have explained. The smallest *fund* of this kind is, indeed, *omnipotent*, if it is allowed time to operate. But we are, I fear, got so near to the limits of the resources of the nation, that it cannot be allowed much time: And, in order to make amends for this, it is necessary that it should be *large*.

In redeeming the public debts, considerable advantages might be derived from their low price; and this advantage, in a time of war, would be nearly the same in whatever manner the redemption was made: That is, whether it was always made at *par*, or at the market prices below *par*. The latter method would be the best; but supposing the former adopted, an equivalent advantage would arise from the power it would give to borrow money on lower terms.—Suppose, for instance, that in a time of war, four millions must be borrowed for the service of the year; and let the produce of the fund be then increased to two millions; and the interest of money in the *stocks*, above 4 *per cent.* In these circumstances, it would be the interest of the lenders of money, to take 3½ *per cent.* for the sums they advanced, in consideration of having their 3 *per cents.* paid off at *par*, to the amount of half these sums.—War, therefore, would accelerate the redemption of the public debts; and it would do this the more the longer it lasted, and the higher it raised the interest of money. Or if, in consequence of paying always at *par*, this could not happen; an equivalent effect would be produced in the way just mentioned. The stocks would be always kept up by the operations of the fund; and, in proportion to the sums yielded by it, the public would be able to borrow money more advantageously, and less would be added to its burdens.—This seems to me an observation of particular consequence. It demonstrates, that the invariable application, in *war* as well as *peace*, of the produce of the fund I am supposing, to the payment of the national debts, rather than to any current services, would, independently of its effect in redeeming these debts, be attended with great advantages to the public. But this is a subject on which I shall have occasion to say more presently.

The *sinking fund*, in its present state, and, after supplying the deficiencies of the peace establishment, yields, I suppose, a considerable part of a million and a half *per ann.*[*] And I cannot doubt but that such savings might be made in the collection and expenditure of the national revenue, as would cause this fund to yield the *whole* of this sum, without imposing any new burdens on the public. But, were there, indeed, no way of providing any part of it, but by creating new funds, or imposing new taxes; it *ought* to be done, because it *must* be done, or the nation sink.

The evils and dangers, attending an *exorbitant* public debt in this country, are so great, that they cannot be exaggerated.—Without repeating, what has been so often said, of its increasing the dependance on the crown; rendering us tributary to foreigners; and raising the price of provisions and labour; and, consequently, checking population, and loading our trade and manufactures; I will only take notice of the following evils which attend it.

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[*] A more careful enquiry has shewn me that the true surplus of the public income was at the time when this was written much less than I imagined. For five years after 1763 it was no more than £300,000 *per ann.* For five years before 1775 it was £338,729; and the medium for the whole duration of the peace between the last two wars was £196,000 *per ann.* according to the accounts in the second Tract on Civil Liberty, Part 3d, Sect. 4th. And from the commencement of the present peace to the present year 1790, the stated revenue has produced *no* surplus.

First. The execrable practices of the *Alley*. These cannot be mentioned in language too strong. They are increasing every day; and the national debt, by giving occasion to them, is likely soon (with the aid of annual lotteries) to ruin all honest industry among us, and to turn us into a nation of gamblers.

Secondly. It must check the exertions of the spirit of liberty in the kingdom. The tendency of every government is to despotism; and in this it must end, if the people are not constantly jealous and watchful. Opposition, therefore, and resistance, are often necessary. But they may throw things into confusion, and occasion the ruin of the public funds. The apprehension of this must influence all who have their interest connected with the preservation of the funds, and incline them always to acquiescence and servility.

But further. It exposes us to particular danger from *foreign* enemies by inviting insults, and rendering us incapable of meeting war, however necessary, without the hazard of bringing on terrible convulsions, by overwhelming public credit.

All these are evils, which must increase with every increase of the national debt; and there is a point at which, when they arrive, the consequence must be fatal

But to proceed to some further observations.

What has been said, has all along supposed a *sacred* and *inviolable* application of the fund I have described, and of all its earnings, to the purpose of sinking the national debt. The whole effect of it depends on its being allowed to operate, WITHOUT INTERRUPTION, a proper time. But it may be asked, how this can be secured? Or, by what method an object, that must be continually growing more and more tempting, can be defended against invasion and rapine?—I might here mention the superintendency and care of the representatives of the kingdom;

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[r] “Either the nation (Mr. *Hume* says, Essays, Vol. II. “p. 145) must destroy public credit; or public credit will “destroy the nation.” Mr. *Gordon*, in the Preface to Cato's Letters, tells us, that the great and good Mr. TRENCHARD had two things much at heart, namely, keeping *England* free from foreign broils, and *paying off the public debts*. He thought that one of these depended on the other, and that the *being* of the state depended on the latter. Mr. *Gordon* adds, that he believed no one, who thought at all, could think Mr. TRENCHARD mistaken.

the faithful guardians of the state, to whom ministers are responsible for the use they make of the public money. But experience has shewn, that we cannot rely on this security.—The difficulty, therefore, now mentioned, is the very greatest difficulty the nation has to struggle with in the payment of its debts.

The sinking fund was established in the year 1716, or soon after the accession of the present family, at a time when the public debts, though not much more than a third of what they are now, were thought to be so considerable as to be alarming and dangerous. It was intended as a SACRED DEPOSIT never to be touched; the law which established it declaring, that it was to be applied to the payment of the principal and interest of such national debts and incumbrances, as had been incurred before the 25th of *December* 1716; *and to no other use, intent, or purpose whatever.*—The faith of *parliament*, therefore, as well as the security of the kingdom, seemed to require, that it should be preserved carefully and rigorously from alienation. But, notwithstanding this, it has been *generally* alienated; and the produce of it employed, in helping to defray such current expences as the exigences of the state rendered necessary.

In order to justify this, it has been usual to plead, that when money is wanted, it makes no difference, whether it is taken from hence, or procured by making a new loan. But in truth the difference between these two methods of procuring money is no less than *infinite*.—For, by employing the SINKING FUND in bearing current expences, rather than borrowing *new* money on new funds; the state, in order to avoid giving *simple interest* for money, is made to alienate money, that *must* have otherwise been improved at *compound interest*; and which, in time, would have *necessarily* increased to *any* sum.—Had a faithful use been made from the first, of only one THIRD of the produce of this fund, the greatest part of our present debts would now have been discharged[s].—Can it be possible then to think, without the deepest regret, of that misapplication of this fund, which, with the consent of parliaments always complying, our ministers have practised?

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[s] The principal observations in this Chapter, I have given just as they occurred to my thoughts, without knowing that any of them had been made by other writers. Some proposals and observations of a similar nature, I have since found in an excellent pamphlet published in 1726, entitled, *An Essay on the National Debts of this kingdom, wherein the importance of discharging them is considered, and some general mistakes about the nature and efficacy of the* SINKING FUND *examined and removed. In a Letter to a Member of the House of Commons. Fourth edition.*

SUPPLEMENT.

THE following account of our Public Funds *in general,* and of the SINKING FUND *in particular,* having been, published in another work too incorrectly, I have thought proper to introduce it here as a *Supplement* to the preceding Chapter.

The *British* Funds have been all formed into the four following classes or divisions.—The AGGREGATE FUND; the SOUTH-SEA FUND; the GENERAL FUND; and the SINKING FUND.

The *Aggregate Fund* was established by an act of *Geo.* I. cap. 12. in 1715. It had this name given it, because it consisted of a great variety of taxes and surpluses of taxes which were in that year consolidated, and given as a security for the discharge of the interest and principal of debts due to the Bank of *England,* and some other public debts; and also for the payment of £120,000 *per annum* to the civil list. Into this *fund* were brought the two-thirds and one-half subsidy of tonnage and poundage; half the inland duties on tea and coffee; the house-money granted by the 7th of *Will.* III.; the duty on hops; the duties on low wines, brandy, and *British* spirits; all arrears of land-taxes; all public monies not appropriated; the surpluses of the nine-penny excise, of the five-sevenths of the Bank ninepenny excise, of the revenues in the annuity acts of the 4th, 5th, and 6th of Queen *Anne*, &c. and, by an act of the 1st of *Geo.* III. all the duties constituting the revenue of the civil list.

The *South-Sea Fund* was established, by stat. 3 *Geo.* I. cap. 9. in 1716; and was so called, because appropriated to pay the interest of the *South-Sea* company's capital. It consisted of a duty on candles, and certain imposts on wines, vinegar, tobacco, and *East-India* goods.

The *General Fund* was also established, by stat. 3 *Geo.* I. cap. 7. in 1716, and consisted of a subsidy on goods exported; a tax on hackney-coaches and chairs; duties on soap, hides, stamps, and policies of insurance; £700 per week letter-money; a moiety of the inland duties on tea and coffee; and £39,855 *per annum* out of the hereditary excise on beer for the bankers annuities. All these taxes were appropriated to the discharge of the interest of £7,808,087, (originally £10,000,000,) capital stock of *South-Sea* annuities, together with charges of management.

All that remained of the produce of the taxes thus digested into these three *funds*, after satisfying the charges upon them, was in the same year (or 1716) carried into a fourth *fund*, to which was given the name of the *Sinking Fund*, because appropriated to the purpose of *sinking* the public debts. The words of the act of the 3d of *Geo.* I. which established this *fund*, are, "All the "monies to arise from time to time, as "well of the excess and surplus of an act "made this session for redeeming the *funds* "of the Bank of *England*; and of the ex"cess or surplus by virtue of one other act "made likewise this session for redeeming "the *funds* of the *South-Sea* company; as "also, of the excess or surplus of the du"ties and revenues by this act appropriated "as aforesaid; and the overplus monies of "the said *General Fund* by this act esta"blished, shall be appropriated to the dis"charging the principal and interest of "such national debts as were incurred be"fore the 25th of December, 1716, and "are declared to be national debts; and to "or for no other use, intent, or purpose "whatsoever." The transactions with respect to this fund make a very important part of the history of *Britain*; and furnish us with a striking instance of the depravity and folly which often ruin kingdoms.

Before its establishment there had existed many smaller *funds* of the same nature; that is, such duties or taxes had been provided for paying the interests of particular loans, as afforded surpluses by which the principal itself was to be gradually redeemed. This was the common practice in the reigns of King *William* and Queen *Anne*. Most of the

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public duties were given for terms of years; and at the end of those terms they ceased of course, unless continued for farther terms by new acts of parliament: And, in general, it was provided, when any money was raised, that the principal should be cancelled either by time, as in the case of the sale of long and short annuities, or by the surpluses of the duties charged with the payment of the interest. This was an excellent plan; but it was by no means carried into execution. In the year 1720, most of the long and short annuities were converted into redeemable perpetuities, at the expence of above three millions; and the surpluses of the duties charged with particular loans were often so broken into, by being either charged with new loans before they had cancelled the old, or spent on current services, as to be rendered incapable of answering the end intended by them. In consequence partly of this bad management, the public debts at the accession of the House of *Hanover* were so much increased as to be generally reckoned *insupportable*; and their reduction was made one of the first objects of parliamentary attention. This gave rise, in 1716, to the institution of the *fund* of which we are giving an account, the father of which was (as has been generally said) Sir *Robert Walpole*, but, in reality, Earl *Stanhope*. All the taxes, except the land-tax and six-pence *per* bushel malt-tax[t], were then made perpetual, and distributed into the three *funds* which have been described, the surpluses of which, for ever afterwards, were to be held sacred, and to be applied *inviolably*, according to the words of the act just recited, to the redemption of the national debt.

A considerate person might have suspected, that the same causes which had rendered former partial appropriations ineffectual, would destroy the efficacy of this. There seemed, however, to be reason for hoping the contrary: For,

First, the future happiness and glory of the kingdom were thought to depend[u] on this appropriation; and the law which established it was declared to be a *fundamental* law of the realm.

Secondly, in conformity to these sentiments, the words of this law were made as strong as possible; and, in order to give additional force to it, a repetition of it, in the same words, was inserted in an act of the 5th of *Geo.* I. cap. 3. Particular notice should be taken of these words. They order that all the surpluses of the taxes then made perpetual, shall be applied to the discharge of the public debts, and "to no other use or "intent whatever." When, therefore, a debt had been paid off, the addition arising from that payment to the surpluses (or the annuity disengaged by it) became a part of the *fund*, and, together with it, was to be employed in discharging farther debts. And the same being true of every successive annuity disengaged by every payment, the *fund*, if never misapplied, must necessarily have operated in sinking the public debt, in the same manner that money accumulates, when put out to bear compound interest. And in this way this *fund* did in fact operate for a few years. While in its infancy, it was watched with great care. The improvement and the inviolable application of it was recommended in most of the speeches from the throne, and echoed back in the addresses of the House of Commons. It is particularly observable, that so well did our ministers then understand the nature and importance of this *fund*, that rather than encroach upon it, they frequently borrowed money in order to defray the necessary expences of government. From some publications in 1726 it appears, that some persons had been led to apprehend this zeal of the ministry would not be permanent, because it was not their interest to pay off the public debt, on account of the dependence and influence created by it. In answering this objection, the writers on the side of the court called such an apprehension an *indecent jealousy*, and took upon them to assure the public, "that in no "possible exigence of affairs could our ministers ever approve of or recommend the "alienation of the *sinking fund*." Happy would it have been for *Britain* had this proved true: But in a little time it appeared, that the apprehensions which had been stiled *indecent jealousies*, were too well grounded. Men in power came soon to see, that this *fund* was advancing too fast in its operations, and to change their zeal for it into a resolution to destroy it. This will abundantly appear from the following facts.

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[t] These taxes have been always voted by parliament from year to year; and are, on this account, distinguished by the name of the *annual* taxes; and wholly employed, as far as they will go, in bearing the current expences of every year.

[u] The alarm occasioned about this time by the public debts (which have been since twice doubled) and the eager expectations entertained from this fund, appear remarkably, from a fact just mentioned; or from the conversion (at the expence of *three millions*) of most of the long and short annuities then subsisting into *redeemable* perpetuities, in order to subject them to the operation of this fund.

Charging the income of the *sinking fund* with the payment of the interest of new loans, is an encroachment upon it, no less subversive of its efficacy, than depriving it of gross sums; there being no difference between taking from it the annual interest of a sum, and that sum itself. Between the years 1727 and 1732 several incroachments of this kind had been made upon it; but, being of a less obvious nature, they passed without any particular opposition. The finishing blow was given in the year 1733. In that year, in order to keep the land-tax at one shilling in the pound, it was necessary either to borrow half a million for the current service, or to take half a million from the *sinking fund.* The last method was chosen; and proposed by Sir *Robert Walpole* to the House of Commons. Long and warm debates ensued. A proposal to alienate, in a time of profound peace, a *fund* which the law had made sacred, and the alienation of which *no possible exigence of public affairs* could justify, only for the sake of keeping the land-tax for one year at one shilling in the pound, justly kindled the indignation of the patriotic party. They urged the prohibition of the law, the faith of parliament, and the security of the kingdom. The proposer of the alienation was reminded of his inconsistency and treachery, in endeavouring to beat down that very monument of glory which he had boasted of having erected for himself; and Sir *John Barnard* warned him, that he was drawing upon himself the curses of posterity. But all arguments were vain. The ministry pleaded that the landed interest wanted ease; that there was no occasion for being in a hurry to pay the national debt; and that the circumstances of the kingdom had altered so much since the establishment of the *sinking fund*, that the competition then among the public creditors was, not who should be *first*, but who should be *last* paid. Thus argued, among others, Sir *Robert Walpole*: *His* reasons prevailed; and the House of Commons consented.

The practice of alienating the *sinking fund* having been thus begun, went on of course. In the next year, or 1734, £1,200,000 was taken from it. In 1735, and 1736, it was anticipated and mortgaged.

Thus expired, after an existence of a few years, the *sinking fund*; that sacred blessing (as it was once thought) and the nation's only hope. Could it have escaped, it would long before this time have eased *Britain* of all its debts, and left it safe and happy.

In order to obtain a juster sense of this, let us here compare what it *would* have done had it never been misapplied, with what it *has* done.

Though the act that established it was passed, as already said, in 1716, it did not begin its operation till 1719, when three quarters of a million in old Exchequer bills were paid off with it. The intermediate time had been employed in laying the foundations of this *fund*, and providing an income for it, by a general reduction of the public debts, from an interest of six *per cent.* and other higher interests, to five *per cent.* What made this reduction then practicable was a rapid fall of the interest of money, which begun (in consequence of an increase of trade producing an influx of money)

money) a little before the accession[x]. The means used by government for accomplishing this reduction were, first, the addition (at *Michaelmas* 1717) of the interest of some debts bearing five *per cent.* to the principal, in order to make use of the produce of the taxes which should have paid that interest, in discharging the bankers debt and some other debts bearing six *per cent.*

Secondly, loans at five *per cent.* obtained chiefly from the Bank, and the *South-Sea* company, to pay off such of the public creditors as did not chuse to accept a lower interest than six *per cent.*

After this reduction, the three *funds*, before described, produced a surplus of above half a million *per annum.* In 1727 this surplus was increased to £939,103, and in 1733 (the year when the practice of alienating it begun) it had been increased so much by the redemptions made with it, and by a second reduction of interest in 1727 from five to four *per cent.* that its medium for five years had been £1,212,000 *per annum.* Had it, from the year 1732, been allowed no increase beyond this sum (except from the interest of debts paid by it). and been applied for the first twenty-five years to the payment of debts bearing four *per cent.* interest, and afterwards to the payment of debts bearing three *per cent.* it would (in the year 1781) have completed the redemption of more than one hundred and sixty millions of debt, leaving the public during this whole period, in possession of all the surpluses which have arisen in the revenue beyond £1,212,000, except those produced by redemptions. It is not possible to conceive the beneficial effects with which this would have been attended, or the vigour which would have been all along given to public credit by such a *fund*, and by the prospect it would have given of the total annihilation in a few years of all the public debts, and the disengagement of the taxes supporting them. No person who duly attends to this, and wishes well to *England*, can avoid execrating the policy which first produced, and has since continued, the alienation of the *sinking fund*, and converted an expedient for saving the kingdom, into a supply for extravagance, and a support of corruption and despotism. This, however, is a policy which it may be expected men in power will always use when they can; for few of them (except when well watched) have ever shewn themselves superior to the temptations of power, or virtuous enough to avoid using all means to extend it.

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[x] The *legal* interest of money was reduced in 1714, from six to five *per cent.*

Many schemes of different kinds have been formed for paying the public debts; and certain it is, that nothing can be of more importance. But the nature of things doth not admit of any method of doing this so expeditiously and effectually as an unalienably *sinking fund*; for in such a *fund* (it has been shewn) money is improved at compound interest, and therefore in the most perfect manner. The writers, therefore, who have employed themselves in contriving such schemes might have spared their labour. The best of all schemes has been long known and established, and received all the weight and efficiency which could be given it by the most solemn acts of the legislature. But no legislature can give security against itself. No parliament can do any thing which it may not undo.

We have now seen what the *sinking fund* WOULD have done, had it suited the views of the *British* ministry in 1733 to suffer it to go on with its operations. Let us next compare this with what it *has* done.

In 1737 and 1738, a million of the stock of Bank annuities and two millions of the stock of *South-Sea* annuities were redeemed with it. For twelve years after 1738, it was wholly applied to the current expences of every year. In 1749, the interest of near fifty-eight millions of the public debts was reduced from four to three and a half *per cent.* interest for seven years, and afterwards to three *per cent.* for ever. But notwithstanding the great addition which this *third* reduction of interest made to the *sinking-fund*, no more than three millions of the public debts were redeemed by it, during the interval of peace between the years 1748 and 1756.

By an act of the 25th of *Geo.* II. 1752, a change was made in the *sinking fund*, which it is necessary to mention.

Before this act the *sinking fund* consisted only of the clear surpluses of the aggregate, the general, and the *South-sea* company's *funds*. By the war, which begun in 1740, there was an addition made to the public debts of near thirty-two millions. This occasioned a great increase of taxes; and the practice was, (whenever any new tax produced *less* than the interest with which it was charged, to make good the deficiency out of the *sinking fund*, and afterwards to replace the sum taken from it out of the supplies for the following year. But whenever a tax produced *more* than the charge upon it, the overplus, instead of being carried to the *sinking fund*, was made a part of the supplies for the year. By the act just mentioned, all the *new* taxes, together with all the annuities to the payment of which they had been appropriated, were ordered to be carried into the *sinking fund*, and formed into one general account. Most of the new taxes having proved deficient, this *fund* at first lost more than it gained by the change. But the loss was afterwards more than made up; first, by the saving which was produced by the reduction of interest from three and a half *per cent.* to three *per cent.* in 1757; and, secondly, by the addition, in the same year, of the salt-duties to this *fund*, after they had completed the redemption of a million, with which they had been charged in 1745.

The war which began in 1756[y], added seventy-one millions and a half to the public debts. This produced a new increase of taxes, which (in conformity to the consolidating act just mentioned) were brought to the general *sinking fund* account, together with the annuities or interest with the payment of which they were charged. And it had been, till the last war, the constant practice to carry every new fund or tax, imposed for paying the interest of a loan, into the *sinking fund*; in consequence of which this *fund* gained when the tax happened to produce more, but lost when it produced less than the interest which it had been given to pay. The *sinking fund*, therefore, which, *before* the consolidating act, consisted only of the surpluses of the aggregate, general, and *South-sea* company's *funds*, consisted afterwards of the clear surplus of all the appropriated taxes. There was only one exception; namely, the additional tax upon houses and windows, granted in 1758, towards paying the interest of four millions and a half then borrowed. This tax was never made a part of the *sinking fund*; and, having always proved deficient to the amount of about £43,000 *per annum*, the deficiency was constantly made good by the *sinking fund*, and afterwards replaced from the supplies.

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[y] See second Tract on Civil Liberty, p. 147.

Before the last reduction of the interest of the public debts, the *sinking fund*, having suffered greatly from various encroachments upon it, produced little more than a million *per ann.* But after this reduction, and its increase, by the addition of the salt-duties, it produced near two millions *per ann.* In 1764 it produced at *Michaelmas*, after making good deficiencies, £2,105,000 nearly. For five years after 1764, its average produce, reckoned to *Christmas* in every year was £2,234,780. For five years, ended in 1774, its average produce (after making good the deficiency of the fund in 1758) was £2,610,759. In 1775, it produced £2,917,869. In 1776, £3,166,517. In 1777, it was charged with an annuity of £100,000 *per ann.* to the civil list; and, after paying three quarters of this annuity, and half a year's interest of five millions borrowed in that year, it produced from October 1776 to October 1777, £2,685,669. From October 1777 to October 1778, 1779, and 1780, it produced £2,442,063, £2,267,399, £2,403,017, after paying the said annuity, and also after making good the deficiency of the fund in 1758, and all the deficiencies [1] of the new taxes.

It appears from this detail, that after the peace in 1763 the income of the *sinking fund* increased considerably. The causes of this have been partly the falling in of life-annuities, and the greater productiveness of the taxes occasioned by the increase of luxury. But the principal cause was the falling in of the interest of about ten millions and a half of the public debts which were discharged during the twelve years of peace between 1763 and 1775. This diminution of the public debts was made, not by the *sinking fund*, but by a contribution from the *East India* Company of £400,000 *per ann.* begun in 1768, and continued for five years; by the profits of ten lotteries; by the composition for maintaining *French* prisoners; sale of *French* prizes taken before the declaration of war in 1756; and other extraordinary receipts, amounting in all to above eight millions. This *fund*, therefore, did not pay off more than two millions and a half, the rest of its produce having been employed in bearing the expences of the peace establishment, which, during that period, were not much less than double to what they had been in any former period.

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[1] These deficiencies have had two causes. First, the unproductiveness of some of the new taxes: and, secondly, the commencement of the interests of the loans before the new taxes could be collected, the payment of which interests, therefore, fell on the Sinking Fund.

To the sum just mentioned, add three millions paid off in the peace between 1748 and 1756, and three millions paid off in 1736 and 1737, and it will appear that the whole amount of the public debts paid off by the *sinking fund*, from its first alienation in 1733, to the commencement of the *American* war, is only eight millions and a half.

It has been said, that when money is wanted for defraying public expences, it makes no difference whether it is obtained by diverting the *sinking fund*, or by a new loan. I have taken notice of this fallacy in page 297. Money in a *sinking fund*, if never alienated, is improved, I have shewn, at *compound* interest; but, when procured by a loan, bears only *simple* interest. A nation, therefore, whenever it applies the income of such a *fund* to current expences rather than the redemption of its debts, chuses to lose the benefit of compound interest in order to avoid paying simple interest; and the loss in this case is equal to the difference between the increase of money at compound and simple interest. The following calculation will shew what this difference is.

One penny put out at our Saviour's birth to five *per cent.* compound interest, would, in the year 1791, have increased to a greater sum than would be contained in THREE

HUNDRED MILLIONS of earths, all solid gold. But, if put out to simple interest, it would in the same time, have amounted to no more than SEVEN SHILLINGS AND SIXPENCE. All governments that alienate *funds* destined for reimbursements, chuse to improve money in the *last* rather than the *first* of these ways.

ADDITIONAL OBSERVATIONS,

BY

THE EDITOR.

THE preceding account brings down the history of the Sinking Fund to the time in which the fourth edition of this work was published. It was the intention of Dr. *Price* in the fifth edition to have given an account of the new Sinking Fund which had been established in the year 1786; but, while engaged in preparing that edition, he was seized by a disorder which soon terminated in his death, and put a final period to that and all the other additions with which he meant to improve the present work.—Being anxious in some degree to fulfil his intentions as well as to render justice to his memory, I published in the following year (1792), an account of his writings on the public finances, with a view principally of introducing the subject of the new Sinking, or Consolidated Fund, and of noticing the great, though unacknowledged assistance which the Minister had derived from his councils in the formation and establishment of it.

That pamphlet has long been out of print, and as I am uncertain whether I shall ever publish another edition, I know of no better method of perpetuating my original design in writing it, as well as of supplying, (though very imperfectly), what Dr. *Price*

Dr. *Price* was prevented from accomplishing, than by inserting the following chapter, which contains all that is most important on the subject in that work.

An Account of three different Plans for redeeming the National Debt, which were formed by Dr. Price *in the year* 1786, *and communicated by him at that time to Mr.* Pitt, *the Chancellor of the Exchequer.*

The measures which had been recommended in the King's Speech during the Administration of Lord *Lansdown* were too little regarded by his immediate successors. The same destructive plan of borrowing money was adopted in the year 1784, which had been invariably pursued for the ten preceding years, and a needless addition was made to the capital of the public debt of near five millions more than the sum advanced. When Mr. *Pitt* returned to power, the loans, which were still necessary to discharge the expences of the late war, were conducted upon better principles. The plan of borrowing at a high rate of interest, which had been so often recommended by Dr. *Price* as the best means of facilitating the redemption of the debt, was now adopted, and almost the whole of the Navy, Victualling, and Ordnance arrears were funded in a stock bearing an interest of *five per cent.* In the year 1786 this improvement in the method of borrowing was succeeded by others of greater importance. The public accounts were thrown into new order; the former divisions of the taxes into four classes under the names of the *Aggregate*, the *South Sea*, the *General*, and the *Sinking Funds*, were abolished, and all included in one fund called the *Consolidated Fund.*. The surplus remaining, after deducting from the whole of the taxes thus consolidated all the expences of the public debts and the Civil List, correspond with what was formerly called the *Sinking Fund.* Out of this surplus a million annually was appointed by Parliament to be applied to the gradual discharge of the public debt; and it was only what remained of it after this deduction that was to be disposable for the expences of the peace-establishment.—This was undoubtedly a very important measure, and well deserving the praise which the Minister assumed to himself as the author of it. From a slight perusal, however, of the present and of the other works of Dr. *Price*, it will be readily seen that this subject had long engaged his attention, and that the nation is indebted to him alone for the first idea of the measure, and for the very method in which it has been carried into execution. Had Mr. *Pitt*, therefore, derived his information only from those works, he would have had but little claim to the lofty pretensions which he made on this occasion; but it will appear from the following account that his obligations extended much further, and that the hope which he expressed in the House of Commons of "having his name "inscribed on a Pillar to Public Credit, as its "preserver and restorer," afforded no very equivocal proof that his *vanity* and *ambition* were equal at least to his *merit* and his *gratitude*.

On the 8th of January, 1786, Dr. *Price* received a letter from Mr. *Pitt*, of which the following is an extract:

"DEAR SIR,

"The subject of the papers which I enclose

" will, I am sure, be an apology for the liberty
" I take in troubling you, and in requesting your
" opinion upon them. When you have had sufficient
" leisure to consider them, I should be
" greatly obliged to you, if you will allow me to
" hope for the pleasure of seeing you at any time
" that is convenient to you.—The situation of the
" revenue certainly makes this the time to establish
" an effectual Sinking Fund. The general
" idea of converting the *three per cents.* into a fund
" bearing a higher rate of interest, with a view to
" facilitate redemption, you have on many occasions
" suggested, and particularly in the papers
" you were so good to send me last year. The
" rise of the Stocks has made a material change
" since that period, and I am inclined to think
" something like the plan I now send you may
" be more adapted to the present circumstances.
" There may be, I believe, some inaccuracies in
" the calculations, but not such as to be very
" material. Before I form any decisive opinion,
" I wish to learn your sentiments upon it; and
" shall think myself obliged to you for any improvement
" you can suggest, if you think the
" principle a right one; or for any other proposal
" which from your knowledge of the subject you
" may think preferable."

By the plan inclosed in this letter, which Mr. *Pitt* intended to have established for reducing the national debt, it was proposed that 107⅓ millions of the *three per cents.* should be converted into 74½ millions nearly of *five per cents*, by changing each £100 stock in the former, which bore an interest of 3 *per cent.* into a capital of £69...10*s*. in the latter, bearing an interest of £3...9*s*...6*d*. This would have produced an additional expence of £510,000 *per annum* nearly, which was to have been deducted from the million surplus intended to be annually applied for the purpose of redemption; and the remaining sum of £490,000 was to have been employed in discharging the principal of this *five per cent.* stock.

The whole of this plan seems to be founded on a principle which Dr. *Price* had often publicly enforced, and which he appears to have repeated in his private communications to Mr. *Pitt*. But surely no principle could be more grossly misapplied. If a judgment were to be formed of the efficacy of a conversion of stock in redeeming the public debts from its feeble operation in the present instance, it might well be concluded, that instead of quickening, it retarded the progress of redemption: for a million annually applied in this manner would not have discharged the 74½ millions of *five per cent.* stock in less than 44 years, or in a shorter period than it would have discharged 107⅓ millions of the *three per cents.* at 90. In the year 1786 the *three per cents.* were between 69 and 70, and therefore had this scheme been carried into execution, it would have been attended with the same disadvantage to the public as if 20 *per cent.* had been then paid on the principal, more than the market price. In delivering his opinion of this scheme Dr. *Price* very strongly urged his objections to it; and Mr. *Pitt* appears from the following letter to be impressed by his arguments.

Holwood Hill, Jan. 15, 1786.

" DEAR SIR,

" I am much obliged to you for the observa-" tions you have been so good to send me on the

" papers I troubled you with. They appear to
" me on the whole very convincing. I shall
" think myself much obliged to you, if you
" would do me the honour to call any morning
" after Tuesday next in Downing-street, when
" you have done with the papers. And if it
" should be in your power to let me know the
" day before, and to fix an early hour, I can be
" certain of being at leisure. The reason of my
" making this request, is, that I think some points
" may occur which may be better explained in
" conversation than by writing, and I am anxious
" to avail myself to the utmost of your assistance,
" where it may be so material. I am, dear Sir,
" with great truth and esteem,
" Your most obedient and faithful servant,
" W. PITT."

In consequence of the interview which Mr. *Pitt* had requested in this letter; he was at length so far convinced of the weakness and insufficiency of his plan as to throw it entirely aside, and Dr. *Price* was induced to communicate to him the three following plans of his own.

PLAN I.

Million *per ann.* Surplus aided by the falling in of the Temporary Annuities, Life Annuities, Expences of Management, and converting sixty millions of the *three per cents.* into *four per cents.* by providing in the first five years £600,000 *per ann.* for paying the difference of Interest.

FUND.		STOCK REDEEMED. 3 per Cents. redeemed.
Surplus	1,000,000 at 75	1,333,333
Received in Exchange for six millions of *three per cents.* converted into 4 *per cents.* at 18⅓ *per cent.*	1,100,000 at 75	1,466,666
Interest of £1,333,333 and £1,466,666	84,000	
Expence of management at 562½ *per* million	1,575	
Short annuity for ten years lapsed	25,000	
Life Annuities lapsed, at £2000 *per ann.*	2,000	
Management on the Short Annuity	351	
Received for 18 millions of *three per cents.* converted into *four per cents.* at 20 *per cent.*	(3,600,000) at 78	4,615,384

One

FUND.		STOCK REDEEMED.
		3 per Cents. redeemed.
One year from the first payment	1,112,926 at 78	1,426,828
Interest of £4,615,384 and £1,426,828	181,266	
Expence of management	3,611	
Life Annuities lapsed	2,000	
Received for 12 millions of *three per cents.* converted into *four per cents.* at 20 *per cent.*	(2,400,000) at 81	2,962,962
Two years from the first payment	1,299,803 at 81	1,604,695
Interest of £2,962,962 and £1,604,695	137,029	
Expence of management	2,568	
Life Annuities lapsed	2,000	
		4 per Cents. redeemed.
Received for 12 millions of *three per cents.* converted into *four per cents.* at 20 *per cent.*	(2,400,000) at par.	2,400,000
Three years	1,441,400	1,441,400
Interest of £2,400,000 and £1,441,400	153,656	
Expence of management	2,160	
Life Annuities lapsed	2,000	
Received for 12 millions of *three per cents.* converted in *four per cents.* at 20 *per cents.*	(2,400,000)	2,400,000

FUND.		STOCK REDEEMED.
		4 per Cents. redeemed.
	£	£
Four years	1,599,216	1,599,216
Interest of £2,400,000 and £1,599,216	159,968	
Expence of management	2,250	
Life Annuities lapsed	2,000	
Exchequer Annuities lapsed ..	56,250	
Five years	1,819,684	1,819,684
Interest of £1,819,684	72,787	
Expence of management	1,023	
Life Annuities lapsed	2,000	
Six years	1,895,494	1,895,494
Interest of £1,895,494	75,820	
Expence of management	1,066	
Life Annuities lapsed	2,000	
		5 per Cents. redeemed at par.
Seven years	1,974,380	1,974,380
&c.	&c.	

	FREE REVENUE.	STOCKS REDEEMED.	
In 25 years after the first payment	£	£	
	5,190,980	13,409,869	three per cents.
		52,959,990	four per cents.
		* 19,953,649	five per cents.
	Total	£86,323,508	
In 30 years after the first payment	6,345,566	13,409,869	three per cents.
		81,128,473	four per cents.
		19,953,649	five per cents.
	Total	£114,491,991	
In 35 years after the first payment	7,511,358	35,633,625	three per cents.
		93,074,916	four per cents.
		19,953,649	five per cents.
	Total	£148,662,190	
In 40 years after the first payment	8,731,523	75,557,308	three per cents.
		93,074,916	four per cents.
		19,953,649	five per cents.
	Total	£188,585,873	

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* In this plan 25 millions of stock are paid off in seven years. The *five per cents.* therefore become redeemable at par, and the whole debt in that stock is discharged at the end of 15 years. From that period the payment of the *four per cents.* again commences, and the whole of them is paid off in the 32d year, from which time the *three per cents.* continue to be redeemed at par during the remainder of the term.

PLAN II.

Million *per annum* Surplus, aided by the falling in of the Temporary Annuities, Life Annuities, Expence of Management, and £600,000 *per ann.* added to the Million Surplus by new Taxes, or Savings, in the first five Years without Conversions.

FUND.			STOCK REDEEMED.
	£	3 per Cents. redeemed.	£
Surplus	1,000,000,	at 75	1,333,333
Interest	40,000		
Management	750		
Annuities lapsed	25,000		
Management on those Annuities	351		
Life Annuities lapsed	2,000		
One year	1,068,101,	at 78	1,369,360
Interest	41,081		
Management	770		
Life Annuities lapsed	2,000		
Additional tax*	60,000		

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* The additional taxes have been introduced into this Table a year too late, by which the Fund loses the interest of £600,000 for a year, that is, £24,000; which would have made the free revenue, after 40 years, about £4000 more than it is by this Table, and the total of Stock redeemed in the same time about £100,000 greater.

Two

	FUND.	STOCK REDEEMED.
	3 per Cents. redeemed.	
	£	£
Two years	1,171,952, at 81	1,446,854
Interest	43,405	
Management	814	
Life Annuities lapsed	2,000	
Additional tax	180,000	
	4 per Cents. redeemed.	
Three years	1,398,171, at par	1,398,171
Interest	55,927	
Management	786	
Life Annuities lapsed	2,000	
Additional tax	120,000	
Four years	1,576,884	1,576,884
Interest	63,075	
Management	887	
Life Annuities lapsed	2,000	
Exchequer Annuities lapsed	56,250	
Additional tax	120,000	
Five years	1,819,096	1,819,096
Interest	72,763	
Management	1023	
Life Annuities lapsed	2,000	
Additional tax	120,000	
Six years	2,014,882	2,014,882
Interest	80,595	
Management	1,123	
Life Annuities lapsed	2,000	
Seven years	2,098,600	2,098,600
&c.	&c.	&c.

In

	FREE REVENUE.	STOCKS REDEEMED.	
	£	£	
In 25 years..	. 5,212,370 ..	21,258,225	three per cents.
		32,769,418	four per cents.
		* 19,968,691	five per cents.
	Total	£73,996,334	
In 30 years....	6,069,709 ..	48,983,157	three per cents.
		32,769,418	four per cents.
		19,968,691	five per cents.
	Total	£101,721,266	
In 35 years	7,060,130 ..	81,258,874	three per cents.
		32,769,418	four per cents.
		19,968,691	five per cents.
	Total	£133,996,983	
In 40 years	8,206,994 ..	118,784,226	three per cents.
		32,769,418	four per cents.
		19,968,691	five per cents.
	Total	£171,522,335	

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* At the expiration of 13 years £20,780,355 in the *four per cents.* and 4,974,151 Stock in the *three per cents.* are paid off. From this period, therefore, the *five per cents.* become redeemable. These are all discharged the beginning of the 20th year, and in two years more the remainder of the *four per cents.* is redeemed. From the 22d year the *three per cents.* are paid off at par, to the end of the term.

PLAN

PLAN III.

Million *per annum* Surplus, aided only by the falling in of the Temporary Annuities, Life Annuities, and Expences of Management.

FUND.		STOCK REDEEMED.
	£	3 per Cents. redeemed.
Surplus	1,000,000, at 75	1,333,333
Interest	40,000	
Management*	750	
Ten Years Annuities lapsed	25,000	
Life Annuities lapsed	2,000	
One year	1,067,750, at 78	1,368,910
Interest	41,067	
Management	770	
Life Annuities lapsed	2000	
Two years	1,111,587, at 81	1,372,329
Interest	41,170	
Management	772	
Life Annuities lapsed	2,000	

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* Since these Tables were computed, the expences of management have been reduced to £450 per million *per ann.*; but the difference this makes in the results is too inconsiderable to deserve notice.

Three

FUND.		STOCK REDEEMED.
	4 per Cents. redeemed.	
	£	£
Three years	1,155,529, at par	1,155,529
Interest	46,221	
Management	650	
Life Annuities lapsed	2,000	
Four years *	1,204,400	1,204,400
Interest	48,176	
Management	677	
Life Annuities lapsed	2000	
Exchequer Annuities lapsed	56,250	
Five years	1,311,503	1,311,503
Interest	52,460	
Management	738	
Life Annuities lapsed	2,000	
Six years	1,366,701	1,366,701
Interest	54,668	
Management	769	
Life Annuities lapsed	2,000	
Seven years	1,424,138	1,424,138
&c.	&c.	&c.

* All the redemptions in these plans, after the first three years, are supposed to be made at par. Exchequer Annuities, amounting to £80,000, fall to the fund in the 20th year, and £404,332 *per ann.* (being the 30 years annuity) in the 22d year.

FREE

	FREE REVENUE.	STOCKS REDEEMED.	
	£	£	
In 25 years	3,892,831 ..	4,074,572	three per cents.
		30,390,243	four per cents.
		18,039,875	five per cents.
	Total	£52,504,690	
In 30 years	4,579,731 ..	21,056,177	three per cents.
		34,283,074	four per cents.
		18,039,875	five per cents.
	Total	£73,379,126	
In 35 years	5,330,252 ..	45,416,595	three per cents.
		34,283,074	four per cents.
		18,039,875	five per cents.
	Total	£97,739,544	
In 40 years ...	. 6,196,104 .	73,747,452	three per cents.
		34,283,074	four per cents.
		18,039,875	five per cents.
	Total	£126,070,401	

EXPLANATIONS AND REMARKS,

BY

DR. PRICE.

"IN the first of these plans the *three per cents.* are supposed at 75, and the difference of price between them and new *four per cents.* supposed to be made *first* redeemable when under *par,* but *ir*redeemable when above *par* till 60 millions of other stocks were redeemed, is reckoned at 18⅓.—The market difference of price between the *present* four per cents. redeemable at pleasure, and the South Sea three per cents. when this plan was formed (the latter being then at 68) was above 20. Was it not reasonable, therefore, to expect that the holders of these annuities would eagerly have taken at 18⅓ *per cent.* an exchange of them for such new four per cents. as those just described, and therefore much more valuable?"

"£18...6*s.*..8*d.* for every £100 three per cent. stock converted into a four per cent. stock would have produced just £1,100,000 for the conversion of six millions. £20 for the conversion of every £100 stock would have produced £1,200,000 for the conversion of six millions."

"This conversion was taken in the first plan at 18⅓ *per cent.* the first year, and only six millions were supposed to be converted, and £1,100,000 then gained for the fund; because the first year was supposed to be a year of experiment only, and therefore more liberal terms were supposed to be offered.—Should the experiment have succeeded, and the difference of price between the three per cents. and the new four per cents. have risen above £20, the former being above 75, a conversion might have been offered the second year, at 20 *per cent.* of 18 millions into four per cents. not redeemable till a larger amount of stocks should have been previously redeemed; and afterwards in the three following years, a third, a fourth, and a fifth conversion, at the same price of twelve millions each year as specified in this plan, the amount of stocks to be previously redeemed to be increased every successive year; in consequence of which the last 12 millions would, without any effect on the efficiency of the plan, have come to be declared the last redeemable of 108 millions."

"Should the Stocks have risen, as is supposed in these plans, greater sums than these specified would probably have been gained by the proposed conversions, and therefore the results in the first plan increased.—There were advantages, not immediately apparent, which would have been gained by these conversions.—First, the public would have availed itself by them of the want of confidence in a plan of redemption which must more or less have taken place at first. For the less the public confidence had happened to have been, the greater would have been the effect of the regulation proposed in raising the value of the new four per cents. and consequently in gaining money for the fund."

"Secondly, it is particularly worth attention that it would have diminished the loss that might have accrued from an interruption of the scheme, should an interruption have happened after £600,000 *per ann.* had been added to a million surplus for carrying on the scheme. Should, for instance, the scheme have been interrupted in the seventh year of its operation, and after the fund had increased, according to the first Table to £1,974,380 and redeemed 25 millions, and, by the 2d Table, to £2,098,600, but redeemed only eleven millions; the public would have gained, by the measure of conversion, the excess of the interest of 27 millions above 13 millions; and, therefore, the addition of £600,000 *per ann.* to the fund in the first five years (which both the first and second Table suppose) would have been nearly replaced, if the plan in the first Table had been adopted. Two advantages, therefore, would in this way have been obtained.—1st. The advantage of a redemption of fourteen millions more than could have been redeemed by the plan in the 2d Table. And, 2dly, the advantage (should the scheme have ever been resumed) of having had 60 millions of four per cents. to be redeemed by a Sinking Fund, instead of 60 millions of three per cents. and consequently a much more easy and rapid course of redemption."

"At the end of the 40 years it is to be observed that the plan in the first Table would have redeemed 188¼ millions, and the plan in the second Table 171¼ millions; but that even in the first *four* years the first plan would have redeemed in the same time only 5½ millions, and but little more than the plan in the third Table. This pointed out a farther and a very considerable recommendation of the first plan, and that was, the particular vigour with which it would have operated at its outset, when vigour was most wanted, and would consequently have brought the *five per cents.* into a course of redemption in six years, which would not have been done by the second plan in less than 13 years, nor will it be done by the third (which is now carrying into execution) in less than 16 years."

Such were the observations and arguments urged by Dr. *Price* in favour of the first plan. But the Minister chose to adopt the third and the weakest, from an apprehension, if he adopted either of the two other plans, that he should be obliged to load the country, in the course of five years, with new taxes to the amount of half a million. Happy had it been for the nation if he had always entertained the same apprehensions.—But he soon threw off every anxiety of this kind, and in a much shorter term than five years had very little hesitation in creating new taxes to the amount of £800,000 *per annum*, for purposes ill calculated to assist the operations of the Sinking Fund.

From the inspection of this plan it appears that, with the aid of the temporary annuities, the appropriated million will in 26 years increase to four millions *per annum*, and redeem above 56 millions of Stock, and that in 40 years it will increase to more than six millions *per annum*, and redeem 120 millions of Stock.—But it was directed by a clause in the Act of Parliament which established this plan, that the accumulation of the fund should be limited to four millions a year, and consequently that the operation of compound interest should be changed into that of simple interest just at the time when it would have redeemed a greater portion of the debt in the next 14 years than it had done in the 26 preceding years. Compared, however, with what Mr. *Pitt* first intended to have established, it deserves respect; and though enfeebled and mutilated by his alterations, it certainly was capable of having done much in relieving the country from the pressure of its burthens. But the public debts have increased since its establishment in a degree so enormous as to render it now a matter of little consequence what plan the Minister may chuse to adopt for their redemption.

The following statements exhibiting a view of the national debt in the year 1786, when the Consolidated Fund was first established, compared with its amount in 1802, afford but little encouragement to hope, from the direction in which Mr. *Pitt* and his successors have proceeded in their management of the finances, that the redemption of the public debt is likely to be soon accomplished, unless it be by a process whose operations are very different from those of compound interest.

Amount of the FUNDED DEBT in the year 1786.

£		£
131,202	Exchequer Annuities for 8 and 22 years then worth at 5 per cent	1,359,000
25,000	Bank Annuities for 10 years Do.	88,500
82,317	Life Annuities, taken at 10 years purchase	823,170
680,375	Bank Annuities for 76 years, then worth at 5 per cent	13,274,117
404,332	Do. for 24 years Do	5,579,781
1,323,226	Life and Temporary Annuities, then worth	21,124,568
	Stock in the 3 per cent. Bank and South Sea Annuities	187,611,254
	Ditto in the 4 per cent. Consolidated Bank Annuities	32,750,000
	Ditto in the 5 per cent. Bank Annuities	17,869,993
	Whole capital of the Funded Debt	£259,355,815

The annual Interest and other charges payable on account of this debt was stated in the Report from the Select Committee of the House of Commons, to be £9,266,940.

Amount of the FUNDED DEBT in April 1802.

		£
£80,223 per ann. Exchequer Annuities, expiring in 3, 4, and 5 years, worth in present money*		207,002
Life Annuities granted chiefly before the present reign, and amounting to £75,492 *per annum*, valued at 7 years purchase ...		528,444
Annuities with benefit of Survivorship amounting to £540 *per annum*		18,000
£123,039 per annum Bank Annuities, expiring in 1808, now worth ..		2,236,900
£1,001,555 per annum Ditto, expiring in 1860, now worth		22,521,300
Present value of Life and Temporary Ann^s.		25,511,646
Stock in the *three per cent.* Perpetual Annuities†		466,674,587
.... *four per cent.* Ditto		49,425,085
.... *five per cent.* Ditto		50,478,039
Borrowed on a Tontine in 1789		1,002,099
		593,091,456
Deduct the Stock redeemed by the Sinking Fund		59,588,904
		533,502,552
Imperial Loans, Stock in the 3 per cents. unredeemed	7,193,906	
£230,000 per ann. for 18 years worth	2,931,120	
	...	10,125,026
Unredeemed Funded Debt of *Great Britain*		£543,627,578

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* The values of these and the following annuities have been computed at *four per cent.* and on the supposition that they are payable *half-yearly.*

† Of this Stock £18,001,148 are stated to be redeemed by the sale of the Land Tax; but while that tax is annually taken in the "Ways and Means" at the usual sum of two millions, it would be as absurd to deduct the stock thus redeemed from the amount of the public debt, as it would be for a person who had sold his estate to continue reckoning the produce of it as a part of his yearly income. It has, indeed, of late been the custom to designate this tax by a different appellation, and to call it a "tax of 4*s.* in the pound on pensions, places, &c." leaving the word *land* entirely out of the account. But such subterfuges are too contemptible even for those who have recourse to them.

In the year 1716, when Sir *Robert Walpole* undertook to redeem the public debts, they did not much exceed the sum of 46 millions: But during the course of 22 years accumulation and redemption proceeded with such equal pace, that in December 1738, when the produce of his Sinking Fund was wholly alienated, those debts amounted nearly to the same sum with which he commenced his operations. In the year 1786, when Mr. *Pitt* engaged in a similar process, the profusion of his predecessors had so far exceeded their economy, as to increase the public debts to 259 millions; but in less than 16 years from the establishment of *his* fund, although 60 millions nearly had been redeemed in the mean time, those debts had swollen to the enormous sum of 543 millions, and the annual expenditure necessary to provide for them amounted to such a sum as would have been sufficient in two years to discharge the whole debt when the first Sinking Fund was established by Sir *Robert Walpole!*

It should be observed that, in addition to the million appropriated to the redemption of the debt in 1786, the sum of £200,000 was annually voted by Parliament for the like purpose, and that these sums, assisted by the temporary annuities which have expired since that period, had redeemed 40 millions nearly. The remaining 20 millions have been discharged by another Sinking Fund established in the year 1792, agreeable to the plan recommended in the preceding chapter of this work;* the design of which was to provide that "in all future loans, £1 *per cent.* on the capital borrowed should be applied annually, together with the accumulating interest, to the redemption of that capitital." In consequence of the unprecedented expenditure of the last war, the annual amount of that fund, exclusive of the interest on the stock redeemed, exceeds £2,500,000. Nor does this sum, great and enormous as it must appear to be, give an adequate idea of the whole expenditure of that disastrous period. The Minister finding himself unable to proceed any longer on the system of funding, had recourse during the last four years to a variety of expedients for raising the supplies: such as forced loans, voluntary contributions, tripled assessments, and above all to that solid system of finance, the *Tax upon Income*, which was to have discharged the debt almost as soon as it was contracted. But these wretched expedients having all failed, they were successively abandoned, and the Minister was again obliged to have recourse to the old system of funding, which he had so lately declared to have been entirely exhausted. By chusing to confide to those expedients rather than to make effectual provision for the debt, new taxes became necessary for £56,445,000 funded on the credit of the Income Tax. This debt, together with £28,750,000 funded in the year 1802, requiring above *two millions and a half per annum* to be raised merely for interest, it was found impossible to raise a further sum of £350,000 *per annum* for the redemption of the debt. Hence the new Sinking Fund was abandoned, and a debt of 85 millions created without any particular provision for the discharge of it. In order, however, to console the nation on relinquishing a measure for which the Minister claimed so much credit to himself and the country, a new expedient was adopted. In lieu of the two former Sinking Funds a new one was established, which by lessening the means, was said to have the surprising effect of increasing the power of redemption. By this plan all the debt created before and since the year 1786 was consolidated, and the sums which were separately applied to the discharge of the *old* and *new* debt were united in one account and applied to the discharge of the *whole*. The sum of £200,000 annually voted by Parliament in aid of the Sinking Fund was also made a perpetual grant for that purpose, and the accumulation is no longer to cease, as in the former plan, when it has amounted to a given sum, but is to proceed until the entire redemption of the public debt is completed. The assistance, however, which the fund was to have derived from the falling in of the temporary annuities, and the reduction of stock bearing 4 and 5 per cent. to a lower interest is now withholden from it, and the Minister has already anticipated some of these future resources by borrowing money on the reversion of them. But notwithstanding this anticipation—notwithstanding the new Sinking Fund is charged with the payment of 85 millions more than the two former ones, our prospects are said to be highly improved, and it is demonstrated by a long series of computations, that if the nation should be blest with an uninterrupted course of peace, and the management of its finances be as much distinguished for economy as it has hitherto been for the want of it, that the debt will be discharged, and every incumbrance removed about the middle of the present century! If any are soothed by such prospects, let them look back to the events of the last century—let them recollect that 40 years of it have been consumed in war, and that more than 540 millions have been added during that disastrous period to the permanent debt of the country. With such woeful experience of the past, is it possible to look forward with confidence, or even with hope to the future? While the same passions prevail among the governments of the world, must not the same consequences inevitably follow? Can any person believe that the present century will be exempt from the calamities of the last; or that in the war in which we are now engaged, the enormity of the taxes will not render it necessary to *alleviate the public burthens** by making fresh inroads into the Sinking Fund?—Perhaps the Minister will, like his predecessor on a former occasion, console the country by observing that "a few years in the existence of a state are not to be counted, and therefore that it is of little consequence whether the bright prospect of redemption shines upon the *middle* or the *close* of this century."—The *present* relief produced by the alienation of the fund will give force to these delusions, and in proportion as its operations are enfeebled and prolonged, the public will become less anxious about it; until, losing all concern for the interest of a remote posterity, and glad to procure even a temporary relief from the growing pressure of their burthens, they abandon it altogether, without any other prospect of the debts being terminated than by the utter extinction of the credit and resources of the country. M.

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* See page 285.

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* See the preamble to the Act for establishing the new Sinking Fund.

NOTES.

NOTES.

Note (A). See Question III. Page 11.

LET E be any given expectation of Life; and $\frac{4E - x}{4E} \times px$ will be the number of persons alive at the end of x years, arising from p persons left annually as widows (or added annually to a town or society) at the age whose *expectation* is E. The *maximum*, therefore, is always pE. In Mr. *De Moivre's Hypothesis*, E is always ½ the difference between the given age and 86. See the Note, p. 2, and the latter end of the Note in p. 39. See likewise the beginning of the First Essay, in Vol. II. and Note (K) in the following Notes, where the investigation of this rule will be given.

It will not be amiss to give the following example of the application of this rule.

At the time of the commencement of the scheme among the ministers and professors in SCOTLAND for making provision for their widows, it was necessary, that a calculation should be made of the number of widows that would be upon the scheme at the end of every year till they came to a *maximum*, on the supposition that, (agreeably to what particular enquiry had shewn to have happened for many preceding years,) 20 new widows would be left every year[*]. In order to make this calculation, let 4 of the 20 widows be supposed to be under 32 years of age when left; and let 28 be supposed their mean age. Let the same number be left between 32 and 39, and let 35 be their mean age; between 39 and 47, and 43 their mean age; between 47 and 57, and 52 their mean age; between 57 and the extremity of life, and 63 their mean age. The number in life together to which, in 10 years, 4 widows left annually at the age of 28 will grow, is, by the rule, (E being 29) $\frac{116-10}{116} \times 40$, or 36.55.——The number alive at the end of 20 years, will be $\frac{116-20}{116} \times 80$, or 66.2. At the end of 30 years, the number alive will be 89; of 40 years, 104.82; of 58 years 116.——These numbers, found in the same way, for the 2d class, (E being 25.5,) at the end of 10, 20, 30, 40, and 51 years, will be 36.7—64.31—84.7—97.25—102——For the third Class, (E being 21.5) at the end of 10, 20, 30, 40, and 43 years, 35.34—61.4—78.13—85.6—86——For the 4th class, (E being 17) at the end of 10, 20, 30, and 34 years, 34.11—56.47—67—68——For the 5th class, (E being 11.5) at the end of 10, 20, and 23 years, 31.3—45.2—46——The whole number, therefore, consisting of all the classes, will come to a *maximum* nearly in 58 years; and the totals in life, at the end of 10, 20, 30, 40, 50, and 58 years, will be 173.37—293.58—364.83—401.67—418.

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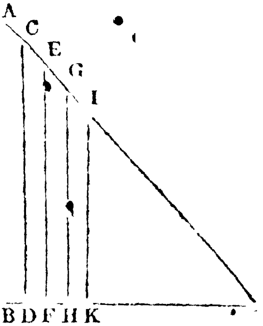
[*] For a term of 35 years and eight months, being from the commencement of the scheme to the year 1783, this number was $19\frac{7}{10}$.

These determinations suppose none to marry. In 10 years, from 1757 to 1767, I have been informed, that but 9 widows married. Let us then suppose, that one widow of the first class marries every year; and let all that marry be supposed to continue, one with another, 5 years in widowhood before they marry. On these suppositions, the foregoing totals will, at the end of the same periods of years, be 169.23—282—347.5—380.17—394.

These calculations are made from Mr. *De*

Moivre's Hyothesis. Had they been made, exactly from Dr. *Halley's* or the *Northampton* Table, the results would have been very nearly the same. See more on this subject in note F.(α)

(α) This theorem is deduced from a fluxional computation in Note (K); but it may be demonstrated without having recourse to fluxions in the following manner: Suppose at the time of admission there were 50 persons aged 36 years, or a number equal to their common complement. Suppose also the same number were added annually at the same age of 36. In the rectangled isosceles triangle ABO let AB (= BO) be = 50, or the complement of a life at 36 —— CD (= DO) be = 49, EF (= FO) be = 48, and so on. Hence BD will be = 1, BF = 2, BH = 3, &c. By the hypothesis of an equal decrement of life it is evident that under the circumstances of this case the area ABCD, or $\frac{2AB - BD}{2} \times BD$ will express the number of annuitants at the end of the first year; the area ABEF, or $\frac{2AB - BF}{2} \times BF$ the number of annuitants at the end of the 2d year; the area ABGH, or $\frac{2AB - BH}{2} \times BH$, the number of annuitants at the end of the 3d year, and so on. If the number of years be x, the annuitants living will be $\frac{2AB - x}{2} \times x$, or $\frac{4E - x}{2} \times x$; for AB being constantly = 50, or the complement, will be twice the expectation, or 2E.——As 50 or 2E, (the numbers of persons admitted annually) is to $\frac{4E - x}{2} \times x$, (the number of annuitants at the end of x years) so is any other number (p) to $\frac{4E - x}{4E} \times px$, the number of annuitants in the same time from (p) persons admitted yearly at the age whose expectation is E; and when x becomes equal to BO (= AB = 2E), the number of annuitants will arrive at its *maximum*, and be constantly expressed by the area pE. Q.E.D. M.



Note (B). Question VI. Page 23.

LET r signify the sum of £1 and its interest, for one year. The value of a life, whose complement is n, being (by Mr. *De Moivre* on *Annuities*, 4th edition, p. 14, and page 100,) $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}+\frac{n-4}{nr^4}$, &c. the present value of the remainder of it after *two* years must be $\frac{n-3}{nr^3}+\frac{n-4}{nr^4}$, &c. which is equal to $\frac{1}{r^2}\times\frac{n-2}{n}\times$ $\frac{n-3}{\overline{n-2}\,r}+\frac{n-4}{\overline{n-2}\,r^2}+\frac{n-5}{\overline{n-2}\,r^3}$, &c.

Now $\frac{1}{r^2}$ is the present value of £1 due at the end of two years. $\frac{n-2}{n}$ is the probability that a life, whose complement is n, shall continue two years, and $\frac{n-3}{\overline{n-2}\,r}+\frac{n-4}{\overline{n-2}\,r^2}+\frac{n-5}{\overline{n-2}\,r^3}$, &c. is the value of a life two years older than the life whose complement is n. And, therefore, (since any number of years less than n may be substituted for two years) the first rule given in this Question is right (β).

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(β) The rules in this and the following Notes are demonstrated rather more satisfactorily, and with equal ease and perspicuity, from the real probabilities of life.

Let a represent the number of persons living in the table at the age of A, and b, c, d, e, &c. the number living at the end of the 1st, 2d, 3d, 4th, &c. years from the age of A. Now since the value of an annuity on the life of A is known to be $=\frac{b}{ar}+\frac{c}{ar^2}+\frac{d}{ar^3}$, &c. the value of this annuity after

two

The same process, applied to joint lives, will demonstrate what is said in the *Scholium*.

two years on the same life will be $= \frac{d}{ar^3} + \frac{e}{ar^4} + \frac{f}{ar^5}$, &c. $= \frac{1}{r^2} \times \frac{c}{a} \times \overline{\frac{d}{cr} + \frac{e}{cr^2} + \frac{f}{cr^3}, + \&c.}$ But $\frac{c}{a}$ is the probability that A lives two years, and the series $\frac{d}{cr} + \frac{e}{cr^2} + \frac{f}{cr^3}$, &c. is the value of an annuity on a life two years older than A. The general rule therefore in the 6th Question is right; for the reasoning applied to this particular case will also apply to any other interval between the present time and the period at which the annuity is to commence. M.

Note (C). See Question VII. Page 25.

LET the complements of any two assigned lives be n and m. The present value of the first possible payment of an annuity to be enjoyed by the life whose complement is n, provided *both* lives continue 7 years, and the life, whose complement is n survives the other *after* that term, is the probability, that the life of the expectant shall continue 8 years, and the other life 7 years and then fail in the 8th year, multiplied by $\frac{1}{r^8}$, or by £1 discounted for 8 years.—The probability that the life of the *expectant* shall continue 8 years is $\frac{n-8}{n}$. The probability that the *other* life shall continue 7 years is $\frac{m-7}{m}$. The probability that it shall continue 7 years, and fail in the 8th year is $\frac{m-7}{m} \times \overline{1 - \frac{m-8}{m-7}} = \frac{1}{m}$. The probability, therefore, that the life of the *expectant* shall continue 8 years, and the other life continue 7 years and fail in the 8th, is $\frac{n-8}{n} \times \frac{1}{m}$; and the present value of the first possible payment of the annuity supposed, is $\frac{n-8}{nr^8} \times \frac{1}{m}$. See *The Doctrine of Annuities*, by Mr. *Simpson*, p. 6—15, or his *Select Exercises*, p. 315, &c.——In like manner, the present value of the 2d payment,

ment, at the end of the 9th year, may be found to be $\frac{n-9}{nr^9} \times \frac{m-7}{m} \times \overline{1 - \frac{m-9}{m-7}}$, or $\frac{n-9}{nr^9} \times \frac{2}{m}$; and the present value of all the possible payments,

$$\frac{1}{r^7} \times \overline{\frac{n-8}{nr} \times \frac{1}{m} + \frac{n-9}{nr^2} \times \frac{2}{m} + \frac{n-10}{nr^3} \times \frac{3}{m}, \&c.}$$

But this series is equal to $\frac{1}{r^7} \times \frac{n-7}{n} \times \frac{m-7}{m} \times$

$$\overline{\frac{n-8}{\overline{n-7}\,r} \times \frac{1}{m-7} + \frac{n-9}{\overline{n-7}\,r^2} \times \frac{2}{m-7} + \frac{n-10}{\overline{n-7}\,r^3} \times \frac{3}{m-7}, \&c.}$$

Now $\frac{n-8}{\overline{n-7}\,r} \times \frac{1}{m-7} + \frac{n-9}{\overline{n-7}\,r^2} \times \frac{2}{m-7}$, &c. is the value of an annuity for a life seven years older than the expectant, after another life seven years older than the life whose complement is m. $\frac{n-7}{n} \times \frac{m-7}{m}$ is the probability that both the assigned lives shall continue 7 years. And $\frac{1}{r^7}$ is the value of £1 due at the end of 7 years. The rule, therefore, given for solving this question, is right.

This demonstration, as well as that in the last note, is, for the sake of more ease and clearness, applied to the hypothesis of an equal decrement of life. It does not, however, depend upon it, but may be applied to any table of observations(γ).

(γ) Let a, b, c, d, e, f, &c. represent the same quantities as in the preceding Note. Let m represent the number of persons living at the age of B, and n, o, p, q, s, &c. the number living at the end of the 1st, 2d, 3d, &c. years from the age of B.—By reasoning in the same manner with Dr. *Price* in the solution of this question, and supposing the first payment of the annuity to become due at the end of the 4th year, its present value will be $= \frac{c}{a} \times \frac{p-q}{mr^4} = \frac{dp}{amr^3} \times \overline{\frac{c}{dr} - \frac{eq}{dpr}}$ —— the present value of the payment at the end of the 5th year will be $= \frac{f}{a} \times \frac{p-s}{mr^5} = \frac{dp}{amr^3} \times \overline{\frac{f}{dr^2} - \frac{sf}{dpr^2}}$ —— the present value of the payment at the end of the 6th year will be $= \frac{g}{a} \times \frac{p-t}{mr^6} = \frac{dp}{amr^3} \times \overline{\frac{g}{dr^3} - \frac{gt}{dpr^3}}$, and so on.

Hence the whole value will be $= \frac{dp}{amr^3} \times \overline{\frac{e}{dr} + \frac{f}{dr^2} + \frac{g}{dr^3}, \&c.}$ $- \frac{dp}{amr^3} \times \overline{\frac{eq}{dpr} + \frac{sf}{dpr^2} + \frac{gt}{dpr^3} + \&c.}$ Let $\overset{\prime}{A}$ and $\overset{\prime}{A}\overset{\prime}{B}$ denote the respective values of annuities on the single and joint lives of two persons 3 years older than A and B, and the general value will become $= \frac{dp}{amr^3} \times \overline{\overset{\prime}{A} - \overset{\prime}{A}\overset{\prime}{B}}$ —— Q. E. D.

M.

Note (D). Question IX. Page 32.

LET the complement of any two assigned lives be *n* and *m*, and the given term be *seven* years, as in Note (C). The probability that the former life (supposed to be the life in expectation) shall last 8 years, is, by Mr. *De Moivre's* Hypothesis, $\frac{n-8}{n}$; and the probability that the latter life shall fail in 8 years, is $\frac{8}{m}$; and the first payment of the annuity mentioned in this question, depends on the happening of *both* these events, the probability of which is $\frac{n-8}{n} \times \frac{8}{m}$.

The present value, therefore, of the first possible payment of the annuity is $\frac{n-8}{nr^8} \times \frac{8}{m}$.——In like manner, the present value of the *second* possible payment is $\frac{n-9}{nr^9} \times \frac{9}{m}$; and of all the payments, $\frac{n-8}{nr^8} \times \frac{8}{m} + \frac{n-9}{nr^9} \times \frac{9}{m} + \frac{n-10}{nr^{10}} \times \frac{10}{m}$, &c. But $\frac{n-8}{nr^8} \times \frac{8}{m} = \frac{n-8}{nr^8} \times \frac{1}{m} + \frac{n-8}{nr^8} \times \frac{7}{m}$; and $\frac{n-9}{nr^9} \times \frac{9}{m} = \frac{n-9}{nr^9} \times \frac{2}{m} + \frac{n-9}{nr^9} + \frac{7}{m}$. The foregoing series, therefore, is equal to the two series's $\frac{1}{r^7} \times \overline{\frac{n-8}{nr} \times \frac{1}{m} + \frac{n-9}{nr^2} \times \frac{2}{m} + \frac{n-10}{nr} \times \frac{3}{m}}$, &c. and $\frac{1}{r^7}$

$$\times\overline{\frac{n-8}{nr}\times\frac{7}{m}+\frac{n-9}{nr^2}\times\frac{7}{m}+\frac{n-10}{nr^3}\times\frac{7}{m}},\text{ \&c. or to}$$

$$\frac{1}{r^7}\times\frac{n-7}{n}\times\frac{m-7}{m}\times\overline{\frac{n-8}{n-7r}\times\frac{1}{m-7}+\frac{n-9}{n-7r^2}\times}$$

$$\overline{\frac{2}{m-7}+\frac{n-10}{n-7r^3}\times\frac{3}{m-7}},\text{ \&c.}+\frac{1}{r^7}\times\frac{7}{m}\times\frac{n-7}{n}\times$$

$$\overline{\frac{n-8}{n-7r}+\frac{n-9}{n-7r^2}+\frac{n-10}{n-7r^3}},$$ &c. which is the very rule given for solving this question (δ), as will appear from Notes (B) and (C).

(δ) Retaining the same symbols as in the two foregoing Notes, (β) and (γ) and suppoing the first payment of the annuity to become due at the end of the 4th year, the present value of the several payments will be $=\frac{e}{a}\times\frac{m-q}{mr^4}+\frac{e}{a}\times\frac{m-s}{mr^5}+\frac{g}{a}\times\frac{m-t}{mr^6}+\text{\&c.}=\frac{d}{ar^3}\times\overline{\frac{e}{dr}+\frac{f}{dr^2}+\frac{g}{dr^3}+\text{\&c}}$ $-\frac{dp}{amr^3}\times\overline{\frac{eq}{dpr}+\frac{sf}{dpr^2}+\frac{gt}{dpr^3}}+\text{\&c.}=\frac{d}{ar^3}\times\overset{1}{A}-\frac{dp}{amr^3}\times\overset{1}{A}\overset{1}{B}$.—Q. E. D.

M.

Note

Note (E). See the Scholium to Quest. X. Page 36.

ACCORDING to the calculations, the time in which the first yearly payment of a reversionary *annuity* becomes due, is the end of the year in which the event happens that entitles to it, however little or much of the year may then happen to be unelapsed. And this, likewise, is the time when a reversionary *sum* becomes due. Those who know how the calculations of the values of reversions are instituted, must know this. But an annuity, the first payment of which is to be made at the same time with another payment of a sum in hand, sufficient to buy an equal annuity, is worth one year's purchase more than that sum. For instance. Reckoning interest at 4 *per cent.* and r being £1 increased by its interest for a year, or 1.04 $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$, &c. = £25 is the present value of an estate of £1 *per annum* for ever. That is, it is the value of it, supposing the first rent of it is to be paid a year hence.—If the first rent is to be received immediately, or at the same time with another payment of £25, it is worth one year's purchase more, or equivalent, to £26.—I have not found, that any of the writers on annuities and reversions, have attended to this observation. It suggests a correction necessary to be applied to the common solutions of several important problems: particularly to the 21st and 22d in Mr. *Simpson's*, *Treatise on Annuities*, and the 26th, 27th, 32d, 33d, and 40th problems in his *Select Exercises*; and to all other problems of the same kind in other writers. There can be no great occasion for being more explicit. It will not, however, be amiss to add the following demonstration. —— $\frac{1}{n}$ is the present probability that a life whose complement is n will fail in any one assignable year of its duration. $S \times \overline{\frac{1}{nr} + \frac{1}{nr^2} + \frac{1}{nr^3}}$, &c. ($n$), or the present value of £1 per annum for n years, multiplied by $\frac{S}{n}$, is the present value of the sum or legacy denoted by S, payable at the failure of the given life. Therefore, (n being 56; the life 30; interest 4 per cent. $r = 1.04$; the sum £25) the value of the expectation, by Mr. *De Moivre's* hypothesis, is 9.919.

Further. The value of £1 to be received at the end of a year, provided the life whose complement is n fails, is the probability of the failure of the life multiplied by £1 discounted for a year, or $\overline{1 - \frac{n-1}{n}} \times \frac{1}{r}$. In like manner; the value of £1 to be received at the end of two years, if the same life fails in 2 years, is $\overline{1 - \frac{n-2}{n}} \times \frac{1}{r^2}$. And, therefore, the value of all the *possible* payments of an estate or annuity of £1 for ever, to be entered upon after the given life, is $\overline{1 - \frac{n-1}{n}} \times \frac{1}{r} + \overline{1 - \frac{n-2}{n}} \times \frac{1}{r^2} + 1 - \frac{n-3}{n} \times \frac{1}{r^3}$, &c. ($n$) $+ \frac{1}{r^n + 1} +$

$\frac{1}{r^n + 2}$

$\frac{1}{r^{n+2}}$, &c. or $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$, &c. $- \overline{\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}}$, &c. that is, the value of the life subtracted from the perpetuity; or, in this example, £14.684, (the value of a life at 30) subtracted from 25; that is, £10.316. But 10.316 is to 9.919, in the same ratio with 104 to 100, or 26 to 25, agreeably to the rule in the *Scholium* (ε).

(ε) The difference between the values of reversionary *sums* and reversionary *estates* (which was first pointed out in this Note) does not depend on the hypothesis of an equal decrement, but may be as readily demonstrated from the real probabilities of life. Supposing *a*, *b*, *c*, *d*, *e*, &c. to represent the same quantities as in Note (β), the value of the *sum* S, to be received on the death of A, will be properly expressed by the series $\frac{S}{a} \times \overline{\frac{a-b}{r} + \frac{b-c}{r^2} + \frac{c-d}{r^3} + \text{\&c.}} = S \times \overline{\frac{1}{r} + \frac{b}{ar^2} + \frac{c}{ar^3} + \frac{d}{ar^4}\ \text{\&c.}} - S \times \overline{\frac{b}{ar} + \frac{c}{ar^2} + \frac{d}{ar^3} + \text{\&c.}} = S \times \overline{\frac{A+1}{r} - A} = \frac{S.\overline{r-1}}{r} \times \overline{P-A}$.——(P denoting the perpetuity, and A the value of an annuity on the life of A).—But in the case of a an *annuity* or *estate*, the value of the reversion of £1 *per annum* after the death of A will be $= \frac{a-b}{ar} + \frac{a-c}{ar^2} + \frac{a-d}{ar^3} + \text{\&c.}$——$(t) + \frac{1}{r^{t+1}} + \frac{1}{r^{t+2}} + \frac{1}{r^{t+3}} +$ &c. (*t* denoting the number of years between the age of A and that of the last surviving life in the table of observations). The sum of these two series is easily found $= P - A$.——If S represent a sum equal to the perpetuity of £1 *per ann.* or, in other words, if S be taken $= \frac{1}{r-1}$, it will appear that the value of the reversion of an *estate* is to the value of the reversion of an equivalent *sum* as $P - A$ to $\frac{P-A}{r}$, or as *r* to 1, agreeable to what has been observed above.

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Note (F). Quest. XIII. Page 48.

WHEN I here call 48 the mean age of all married men, and 40 the mean age of married women, I do not intend to suppose, that there are as many married persons who exceed these ages, as there are who fall short of them. It is likely that the latter are more numerous; and it is necessary that this should be the case, to render the supposition I make just.—If all marriages commenced at 33 for the man, and 25 for the woman, one half of them would be dissolved by the time the men were 50, and the women 42; for (by the *Hypothesis*, and also nearly by the *Breslaw*, *Norwich*, and *Northampton* tables) there is an equal chance for the joint continuance of two lives whose ages are 25 and 33, *seventeen* years. Forty-two and fifty then would be properly the mean ages at which widowhood would commence: meaning by these "the "ages on each side of which equal numbers are "left widows and widowers." But, though in this case half the marriages of every year would be dissolved in 17 years, they would not be *all* dissolved in twice that time. So far would this be from happening, that about a 7th part would continue beyond twice 17 years; nor would it be *certain*, that they would be all dissolved till near the extremity of the possible extent of life. Though, therefore, an equal number of marriages would be dissolved, or an equal number of widows and widowers left *before* 50 and 42 and *afterwards*, yet the ages of the latter would, one with another, much more exceed 50 and 42, than the ages of the former (that is, of the widows and widowers left *before* 50 and 42) would fall short of them. And the number of marriages also in the world among persons of greater ages than these, would be much fewer than among persons of lesser ages.—In other words: The period, at which the marriages that have been contracted are half dissolved, is not the period at which the number of marriages constantly existing is equally divided, but this period falls some years sooner; and the period I have in view falls in that part of the interval between these two periods, where the greater ages of the marriages on one side, are just enough to compensate (in such a calculation as that I have given) their deficiencies in number, compared with the number of marriages on the other side.

In short. Suppose 35 marriages every year, between persons 33 and 25[b]. In 12 years there would be half as many in the world, as could possibly arise from such a number of yearly weddings. In 17 years, half every set would be extinct. The *expectation* of every marriage would be 19 years, by Prob. 21 of Mr. *De Moivre's Treatise on Annuities*, or by the note (K) in the following notes: That is, taking them all together, they would exist just as long as an equal number of *single* persons, supposed to be sure of living just 19 years, and no more: or, as long as an equal number of single persons, all 48 years of age, supposed to be subject to the common laws of mortality. One with another, then, they will be all extinct in 19 years; the marriages which continue beyond this term, though fewer in number, enjoying among them just as much *more* duration, as those that fall short of it enjoy *less*. *Widows*, then, at a medium, will commence widowhood at 44 (that is, 25 increased by 19) years of age, and *widowers* at 52. The values, therefore of the lives of the *former*, when they commence widowhood, will, one with another, be the same with the value of a life at 44; or, (reckoning interest at 4 per cent.) 12.5 years purchase, in one present payment, (the annuity to begin at the end of a year); and their *expectation* of life will be 21 years, or half the difference between 44 and 86. The value of the lives of the *latter* will be 10.92, and their *expectation* 17 years.—The whole number of marriages constantly existing, which would result from 35 supposed to commence annually, would be 19×35, or 665; and 53 years (the difference between 33 and 86) would be the time in which they would increase to this number—The chance of survivorship would be the odds of 69 to 53, by Prob. 18th, Mr. *De Moivre on Annuities*; that is, in 53 years, 35 relicts of these marriages would be left every year, and the number of *widows* would be to the number of *widowers*, as 69 to 53; or 19.8 *widows* would be left annually, and 15.2 *widowers*. The *maximum* of widows in life together, if none married, would be 21×19.8, or 416; and they would increase to this number in 114 years (or 61 years after the number of marriages had arrived at a *maximum*)—The *maximum* of *widowers* would be 15.2×17, or 258; and they would increase to this number in 106 years.

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[b] In the *Pais de Vaud, Switzerland*, the mean age at which women marry, is nearly the very age here mentioned: But it is shewn in the *Supplement*, that the expectation of marriage there is no less than 23 years and ½; so much higher are the probabilities of life in the *country* than in *towns*, or than they ought to be according to Mr. *De Moivre's Hypotheſis*. See Supplement, Vol. II.

An easy method may be hence deduced of solving the question which occasions this note.—If the number of the members of the establishment I have supposed is 665, and the mean ages at which marriage may be deemed to commence are

25 and 33, 19.8 widows will (it has just appeared) be left every year; and the values of their lives, when they commence widowhood, will be, one with another, 12½ years purchase. An annuity of £20 will, therefore, be worth, to each widow, £250, and 19.8 such annuities must be worth £4950, which, consequently, is the annual income necessary for the support of the establishment, the first payment to be received immediately: or £7.44 from each of the 665 members; which answers nearly to the determination in p. 48.

In the latter part of Essay II. Vol. II. it is shewn, that observations determine the chance of survivorship in favour of the wife in marriage, to be really so great as 3 to 2; and in some circumstances greater. It is there observed, that in order to account for this from the difference of age between men and their wives, this difference must be at least 12 years, and the mean ages of all who marry annually must be supposed to be about 23 and 35. In this case, 19, as before, will nearly be the *expectation* of all marriages. The mean age at which widows and widowers will commence such will be 42 and 54. The number of annual marriages necessary to keep up 665 marriages constantly existing, will be 35. The number of widows left annually, by such a number of marriages, will be 21; and the values of their lives, at the time they commence widowhood, will be 12.85 years purchase by Table VI. Vol. II. and, therefore, the whole annual income necessary for the support of the supposed establishment, will be £5397, or an annual payment, beginning immediately, of £8.11 from each member—The number of widows on such an establishment will, in 63 years, grow, if none marry, to 462; and the number of widowers to 224.

—It may be depended on, that all this would happen as far as Dr. *Halley's* Table, or the Tables for *Norwich* and *Northampton*, exhibit the true state of human mortality.

Among the ministers and professors in SCOTLAND, the number of married men being 667, or nearly that here mentioned, the number of annual weddings has, for many years, been at an average 30, and the number of windows left annually 19¼; and, therefore, the chance of survivorship in favour of the wife, as 19.2 to 11.8, or 5 to 3. This is not more different from the results I have given, than might have been expected; and the chief reason of the difference is, that the *expectations* of *single* and *joint* lives among the ministers and their wives in SCOTLAND are greater than those given by Dr. *Halley's*, and the other tables of observation.—These tables give the expectations of lives as they are among the bulk of mankind in moderate towns. The expectations of lives among the better sort of men, living mostly in country villages and parishes, are much greater. The fact is, that among the ministers in *Scotland*, the expectation of a *single* life, at the age of 27, is near 4 years greater; and, of *joint* lives, about three years greater, than the same expectations by Dr. *Halley's* Table. See the latter end of the 2d Essay in the next volume.

I cannot help just mentioning another remark here.—It may be observed, that supposing no second marriages, and, at the same time, that the odds for the woman's surviving in marriage is 3 to 2, the number of *widows* in the world would be *double* the number of *widowers*. But it has been found, in fact, that the number of widows is, in some situations, five times the number of widowers. How this is to be accounted for, I have shewn in the Essay just referred to.

Note

Note (G). Question XIV. Page 52.

LET r be £1 increased by its interest for one year; t the given time or number of years for which the assurance is to be made; a, b, c, &c. the *probabilities* taken out of a table of observations, that the person whose age is given shall live 1, 2, 3, &c. years; and P the probability that he shall live t years. Then $\overline{\frac{1-a}{r} + \frac{1-b}{r^2} + \frac{1-c}{r^3}\text{, \&c. } (t-1)} + \frac{1-P}{r^t} + \frac{1-P}{r^{t+1}} + \frac{1-P}{r^{t+2}}$, &c. $= \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}\text{, \&c. } (t)} - \overline{\frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3}\text{, \&c. } (t-1)} + \frac{P}{r^t} + \frac{1-P}{r^t} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}}$, &c. will be the exact value of an annuity to be entered upon at the failure of the given life, provided it happens in t years. And the rule is nothing but this value expressed in words. In a similar manner may be demonstrated the other rule for finding the values of assurances for a given time, on two joint lives, or the longest of two lives.*

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* Having given a different and more concise rule for answering this question, (See note, p. 53,) it may not be improper here to insert the demonstration of it.—Let a represent the number living in the table at the age assigned, let b, c, d, e, &c. represent the same number at the end of 1, 2, 3, 4, &c. years; then will the value of the sum S depending on the failure of the life within the term proposed be $= S \times \overline{\frac{a-b}{ar} + \frac{b-c}{ar^2} + \frac{c-d}{ar^3} + \frac{d-e}{ar^4} \text{ - - - - } (t)} = \frac{S}{r} \times \overline{1 + \frac{b}{ar} + \frac{c}{ar^2} + \frac{d}{ar^3} \text{ - - - - } (t-1)} - S \times \overline{\frac{b}{ar} + \frac{c}{ar^2} + \frac{d}{ar^3} + \frac{e}{ar^4} \text{ - - - - } (t)}$. If B′ denote the value of an annuity of £1 on the given life for one year less than the term proposed, and B the same value for the whole term, the preceding series will be $= S \times \overline{\underline{1 + B'}} - B$ - - - Q. E. D.

M.

Note (H.) Question XV. Page 61.

LET r signify as before; S the given sum to be assured; t the given time; N and n the number of the living in the table of observations, at the age of B and A respectively; A, B, C, &c. and a, b, c, &c. the number of the living in the table, at the end of 1, 2, 3, &c. years from the ages of B and A; D, $D_{\prime}$, $D_{\prime\prime}$, $D_{\prime\prime\prime}$ &c. and d, $d_{\prime}$, $d_{\prime\prime}$, $d_{\prime\prime\prime}$, &c. the decrements of life in the table, at the end of 1, 2, 3, &c. years from the same ages. Then, by reasoning in the same manner with Mr. *Simpson*, in p. 316, &c. *Select Exercises*, it will appear that

$$S \times \overline{\frac{A \times d}{Nnr} + \frac{B \times d_{\prime}}{Nnr^{2}} + \frac{C \times d_{\prime\prime}}{Nnr^{3}}, \text{\&c. } (t)} + S \times \overline{\frac{Dd}{2Nnr} + \frac{D_{\prime}d_{\prime}}{2Nnr^{2}} + \frac{D_{\prime\prime}d_{\prime\prime}}{2Nnr^{3}}, \text{\&c. } (t)} = \frac{S}{n} \times \overline{\frac{Ad}{Nr} + \frac{Bd_{\prime}}{Nr^{2}} + \frac{Cd_{\prime\prime}}{Nr^{3}}, \text{\&c. } (t)} + \frac{S}{2N} \times \overline{\frac{Dd}{nr} + \frac{D_{\prime}d_{\prime}}{nr^{2}}, \text{\&c. } (t)}.$$

This is the exact answer to Question XV. and the rule is as near an approximation to it as there is reason to desire.

In the same manner, retaining all the same symbols, it may be found, that the answer to Question XVI. is

$$S \times \overline{\frac{Dd}{2Nnr} + \frac{Dd_{\prime}}{2Nnr^{2}} + \frac{\overline{D + D_{\prime}} \times d_{\prime\prime}}{Nnr^{3}} + \frac{\overline{D + D_{\prime} + D_{\prime\prime}} \times d_{\prime\prime\prime}}{Nnr^{4}} (t)}, \text{\&c. } + S \times \overline{\frac{D_{\prime}d_{\prime}}{1Nnr^{2}} + \frac{D_{\prime\prime}d_{\prime\prime}}{2Nnr^{3}} + \frac{D_{\prime\prime\prime}d_{\prime\prime\prime}}{2Nnr^{4}}, \text{\&c. } (t-1)}.$$

$$(t-1) = \frac{S}{nr} \times \overline{\frac{Dd_{\prime}}{Nr} + \frac{\overline{D + D_{\prime\prime}} \times d_{\prime\prime\prime}}{Nr^2} + \frac{\overline{D + D_{\prime} + D_{\prime\prime}} \times d_{\prime\prime\prime}}{Nr^3}}$$

$$\&c\ (t-1) + \frac{S}{2N} \times \overline{\frac{Dd}{nr} + \frac{D_{\prime}d_{\prime\prime}}{nr^2} + \frac{D_{\prime\prime}d_{\prime\prime\prime}}{nr^3},\ \&c..\ (t)}.$$

But $\frac{D}{Nr} + \frac{D + D_{\prime}}{Nr^2} + \frac{D + D_{\prime} + D_{\prime\prime}}{Nr^3}$, &c. $(t-1)$ is the same with the excess of the value of an annuity *certain* for a number of years less by one year than the given term, above the value of an annuity on the life of A, for the same number of years; from whence the reason of the rule for solving this question may be easily discovered (ζ).

ζ The solution of the 15th question may be deduced in a similar, but rather more accurate, manner from the first of the two rules given in Note (O); where the value of the reversion for t years is expressed by the two series $\frac{S}{2ab} \times \overline{\frac{ca'}{r} + \frac{da''}{r^2} + \frac{ea'''}{r^3} \cdots\cdots (t)} + \frac{S}{2ab} \times \overline{\frac{ba}{r} + \frac{ca''}{r} + \frac{da'''}{r^3} \cdots\cdots (t)}$. If α denote the sum of the decrements of life from the age of A for t years divided by t (which may be called the *complement* of A's life for the given term), and B and $\overset{\prime}{B}$ the values of an annuity on the life of B for t and $\overline{t-1}$ years respectively, the sum of these two series may be found $= \frac{S\alpha}{2a} \times \overline{B + \frac{\overset{\prime}{B} + 1}{r}}$.

In like manner, the solution of the 16th Question may be derived from the second of the two rules given in Note (O);—the series expressing the value of the reversion in this case being $\frac{S}{2ab} \times \overline{\frac{b - c.a'}{r} + \frac{c - d.\overline{a' + a''}}{r^2} \cdots (t)} + \frac{S}{2abr} \times \overline{\frac{c - d.a'}{r} + \frac{d - e.\overline{a' + a''}}{r^2} \cdots (t-1)}$. Let β denote the *complement* of B's life for t years, A and $\dot{A}$ the values of an annuity on the life of A for t and $\overline{t-1}$ years, and N and $\dot{N}$ the values of an annuity certain for those respective terms; then will the above series be found $= \frac{S.\beta}{2b} \times \overline{N - A} + \frac{\dot{N} - \dot{A}}{r}$.——It is to be observed, when the decrements of A's life for t years, in the first of these rules, and the decrements of B's life in the second, are equal, that the exact value of the reversion is obtained; and if the term do not exceed 10 or 12 years, the values are always so nearly true as not to require greater accuracy. This also is the case in general with regard to Dr. *Price's* rules; against which there can be no objection, excepting the application of Mr. *De Moivre's* hypothesis in one part of them, which it is best *entirely* to exclude from the doctrine of survivorships. M.

Note (I). Page 149.

SUPPOSING r to signify as in the last notes, and n to be the complement of a given life; the present value of £1, £2, £3, &c. payable at the end of 1, 2, 3, &c. years to t years, but subject to failure when the life fails, is $\frac{n-1}{nr} + \frac{\overline{n-2}\times 2}{nr^2} + \frac{\overline{n-3}\times 3}{nr^3}$, &c. continued to t years; which expression is equal to $n \times \overline{\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}, \&c. \ (t)}$ $- n \times \overline{\frac{\overline{n-1}^2}{n^2r} + \frac{\overline{n-2}^2}{n^2r^2} + \frac{\overline{n-3}^2}{n^2r^3}, \&c. \ (t)}$.

To find, therefore, the value of an annual payment dependent on a given life, to begin with £1 and to increase at the rate of £1 every year after the first, for a given term; find the value of an annuity on the given life for the given term; and also the value for the given term of an annuity on two joint lives both equal to the given life. The difference between these two values multiplied by the complement of the given life, will be the value sought.—If such a course of payment, instead of beginning at the end of a year, is to begin immediately, and to be made at the beginning of every year till $t + 1$ payments are made in t years; add to the preceding value the value increased by unity of an annuity on the given life for t years, found by Question VI, and the sum will be the value sought. And this value, divided by the present value of what may happen to remain of the given life after t years, found by the same question, will give the annuity to which such a series of increasing annual payments, beginning immediately, will entitle for the remainder of the given life after t years.

If such a course of payment is to begin at the end of a year, and to be continued during life (that is, if $t = n$) it is obvious, that its value will be the complement of the life multiplied by the difference between the value of the life, and the value of two joint lives having the same common age with it; and that if it is not to commence till the end of a given number of years, its value will be the value for a life so many years (lessened by one) older than the given life, and multiplied by the value of £1 payable at the end of a number of years, and also multiplied by the probability that the given life will exist for the same number of years.—Supposing, for instance, the given life 30 years of age, and such a course of payment to begin when it has completed its 56th year, the value would be the value of a life aged 55, diminished by the value of two joint lives both 55, and the remainder multiplied by the complement of a life aged 55, and also by the product of the probability that a life aged 30 will exist 25 years, into the value of £1 payable at the end of 25 years.—The value thus computed will, in this case, come out £19 nearly, in a single present payment, reckoning interest at 4 *per cent.* and taking the probabilities of the duration of life from the *Northampton* Table of Observations.

With the assistance of these rules, all that is said in p. 148 may be investigated. But more particular directions for computing the values of annuities of this sort may be found in Mr. *Morgan's Treatise on Life Annuities and Assurances*, p. 119, &c.

Note (K). See Essay I. Vol. II.

THE *sum* of the probabilities that any given lives will attain to the end of the 1st, 2d, 3d, &c. *years* from the present time to the utmost extremity of life (for instance, $\frac{45}{46}+\frac{44}{46}+\frac{43}{46}$, &c. to $\frac{1}{46}=22\frac{1}{2}$ for lives of 40, by the *hypothesis*) may be called their *expectation*, or the number of payments due to them, as *yearly annuitants*. The sum of the probabilities that they will attain to the end of the 1st, 2d, 3d, &c. *half years*, (or, in the particular case specified, $\frac{91}{92}+\frac{90}{92}+\frac{89}{92}+\frac{88}{92}$, &c. $=\frac{91}{2}$ *half years*, or $22\frac{3}{4}$ *years*) is their expectation as *half-yearly annuitants*. And the sums just mentioned of the probabilities of their attaining to the end of the 1st, 2d, 3d, &c. *moments* (equal in the same particular case to 23 years) is properly their *expectation of life*, or their *expectation* as annuitants secured by land.

Mr. *De Moivre* has omitted the demonstrations of the rules he has given for finding the *expectations* of lives, and only intimated in general, that he discovered them by a calculation deduced from the method of fluxions. See his *Treatise on Annuities*, page 66. It will, perhaps, be agreeable to some to see how easily they are deduced in this method, upon the hypothesis of an equal decrement of life.

Let $\dot{x}$ stand for a moment of time, and n the *complement* of any assigned life. Then $\frac{n-\dot{x}}{n}$, $\frac{n-2\dot{x}}{n}$, $\frac{n-3\dot{x}}{n}$, &c. will be the *present* probabilities of its continuing to the end of the 1st, 2d, 3d, &c. moments; and $\frac{n-x}{n}$ the probability of its continuing to the end of x time. $\frac{n-x}{n} \times \dot{x}$ will therefore be the *fluxion* of the sum of the probabilities, or of an *area* representing this sum, whose *ordinates* are $\frac{n-x}{n}$, and *axis* x.—The *fluent* of this expression, or $x - \frac{x^2}{2n}$, is the sum itself for the time x; and this, when $x = n$, becomes $\frac{1}{2}n$, and gives the *expectation* of the assigned life, or the sum of all the probabilities just mentioned, for its whole possible duration. —In like manner: since $\frac{\overline{n-x}|^2}{n^2}$ is the probability that two equal joint lives will continue x time, $\frac{\overline{n-x}|^2}{n^2} \times \dot{x}$ will be the *fluxion* of the sum of the probabilities. The *fluent* is $x - \frac{x^2}{n} + \frac{x^3}{3n^2}$, which, when $x = n$, is $\frac{n}{3}$, or the expectation of two equal joint lives.— Again: since $\frac{n-x}{n} \times \frac{2x}{n}$ is the probability that there will be a survivor of two equal joint lives at the end of x time, $\frac{n-x}{n} \times \frac{2x}{n} \times \dot{x}$ will be the *fluxion* of the sum of the probabilities; and the *fluent*, or $\frac{x^2}{n} - \frac{2x^3}{3n^2}$ is (when $x = n$) $\frac{1}{3}n$, or the *expectation* of survivorship between two equal lives; which, therefore, appears to be equal to the *expectation* of their joint continuance. The expectation of two *unequal* joint lives, found in the same way, is $\frac{m}{2} - \frac{m^2}{6n}$, m (n) being the *complement* of the oldest life, and n the *complement* of the youngest. The whole expectation of survivorship is $\frac{n}{2} - \frac{m}{2} + \frac{m^2}{3n}$ (9). And the expectation of survivorship of the oldest will be to the expectation of survivorship of the youngest, as $\frac{m^2}{6n}$ to $\frac{n}{2} - \frac{m}{2} + \frac{m^2}{6n}$. It is easy to apply this investigation to any number of joint lives, and to all cases of survivorship.

(n) The expectation of two unequal joint lives is $= \frac{m-x}{m} \times \frac{n-x}{n} \times \dot{x}$, whose *fluent* (when $x = m$) is easily found $= \frac{m}{2} - \frac{mm}{6n}$.

It may be observed, concerning the first of the fluents here given, that it expresses not only the expectation of a given life for the time x, and therefore its whole expectation when $x = m$, but likewise the number of persons alive, to which one person added annually to a society, at a given age, will increase in x time.——Thus: Suppose one annuitant, whose age is 28, (and whose *complement* of life, therefore, is 58, or *expectation* of life 29) to come upon a society every year; the number of annuitants alive, deduced from hence, will in x years, be $x - \frac{x^2}{4\times 29}$, or $\frac{4\times 29 - x}{4\times 29}\times x$; and, therefore, the number of annuitants alive, deduced in the same time from p annuitants left annually at the same age, will be $\frac{4\times 29 - x}{4\times 29}\times px$.—In like manner, the 2d fluent, or $\frac{x^3}{3n^2} - \frac{x^3}{n} + x$, gives the number of marriages in being together, that will, in x years, grow out of *one* yearly marriage, between persons of *equal* ages, whose complement of life is n. If they are of *unequal* ages, and the complement of the oldest life is m, and of the youngest n, this number will be $\frac{x^3}{3nm} - \frac{\overline{n+m}\times x^2}{2nm} + x$. And if the number of years is required, in which any given number of yearly marriages, between men and women at given ages, will increase so far as to be in any given proportion to the greatest number that can possibly grow out of such marriages, this expression must be made equal to the *expectation* of the joint lives, or of each marriage, multiplied by the fraction expressing the given proportion; and the root of the equation will be the answer. Thus: it may be found, that one marriage every year, between persons 33 and 25 years of age, would in 10 years increase to 8.35; in 15 years to 11.38; and in 53 years, to 19, or their greatest possible number; and, consequently, that 35 such yearly marriages would, in 10 years, increase to 292; in 15 years to 398; and in 53 years, to 665.——And if it is enquired in what number of years 35 such yearly marriages would increase to half the number in being together, possible to be derived from them, the value of x, in the cubic equation $\frac{x^3}{3nm} - \frac{\overline{n+m} \times x^2}{2nm} + x = \overline{\frac{m}{2} - \frac{m}{6n}} \times \frac{1}{q}$, must be found; which, in the present instance, is nearly 12.

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(9) The expectation of survivorship due to the oldest life is expressed by the *fluxion* $\frac{m-x}{m} \times \frac{x}{n} \times \dot{x}$, whose *fluent* (when $x = m$) is $\frac{mm}{6n}$. The expectation of survivorship due to the youngest life for m years is the *fluent* of $\frac{n-x}{n} \times \frac{x}{m} \times \dot{x}$; which (when $x = m$) is $\frac{m}{2} - \frac{mm}{3n}$. But this life has a further expectation, after m years, expressed by the *fluent* of $\frac{n-m-x}{n-m} \times \frac{n-m}{n} \times \dot{x}$, which (when $x = n - m$) will be $\frac{n}{2} - m + \frac{mm}{2n}$. The sum of these two *fluents*, or $\frac{n-m}{2} + \frac{mm}{6n}$ will therefore be the whole expectation of survivorship due to the youngest life. And this expression added to $\frac{mm}{6n}$ (which has been found above to be equal to the expectation of survivorship due to the oldest life) will give $\frac{n}{2} - \frac{m}{2} + \frac{mm}{3n}$ for the whole expectation of survivorship due to both lives. M.

I have, in some parts of this work, had occasion to make such deductions as these. See note (A), and note (F); and Questions III. page 10, and XIII. page 46.

Note (L). Chapter IV. Page 208.

LET r signify £1 increased by its interest for one year.

V the PERPETUITY.

n the difference between the age of the youngest life, and 86; or its *complement*.

m the complement of the oldest life.

P the value (in Table II. Vol. II.) of an annuity certain for m years.

And the exact value of any two given joint lives, according to the hypothesis of an equal decrement of life, will be $V - \frac{V+1}{n} \times \overline{n - m - 2V - 1 \times \frac{P}{m} + 2V}$ (*). Example:

Let

(*) This general rule is taken from Mr. *Simpson's* Doctrine of Annuities, and is easily demonstrated by the assistance of the Theorems in the 5th Chapter in this volume, page 257. The series expressing the value of an annuity on two joint lives, whose complements are n and m, is known to be $= \frac{m-1}{m} \times \frac{n-1}{nr} + \frac{m-2}{m} \times \frac{n-2}{nr^2} \text{----} (m) = \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \text{---} (m) - \frac{m+n}{mn} \times \overline{\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} +, \&c. \text{----} (m)} + \frac{1}{mn} \times \overline{\frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} \text{---} (m)}$. By the Theorems just referred to, the first of these series may be found $= \frac{1}{r-1} - \frac{1}{r^m.\overline{r-1}}$ ---- the second $= -\frac{m+n}{mn} \times \frac{r}{\overline{r-1}|^2} + \frac{mm+mn}{mn} \times \frac{1}{r^m.\overline{r-1}} + \frac{m+n}{mn} \times \frac{r}{r^m.\overline{r-1}|^2}$, and the third $= \frac{r^2+r}{m.\overline{r-1}|^3} - \frac{mm}{mnr^m} \times \frac{1}{r-1} + \frac{2mr}{mnr^m} \times \frac{1}{\overline{r-1}|^2} - \frac{r^2+r}{mnr^m} \times \frac{1}{\overline{r-1}|^3}$.—Adding these different series together their sum

will

Let the ages be 27 and 38; and the rate of interest 4 *per cent.* Then $n = 59$. $m = 48$. $V = 23$. $P =$

will be $= \frac{1}{r-1} - \frac{m+n}{mn} \times \frac{r}{\overline{r-1}|^2} + \frac{n-m}{mn} \times \frac{[illegible]}{r^m \overline{r-1}|^2} + \frac{r^2+r}{mn.\overline{r-1}|^2} - \frac{r^2+r}{mn.r.^m\overline{r-1}|^3}$, = (since $\frac{r}{r-1}$ is $= V + 1$, and $\frac{r^2+r}{\overline{r-1}|^3}$, is $= \frac{r}{\overline{r-1}|^2} + \frac{2r}{\overline{r-1}|^3}$,) $V - \frac{m+n-1}{mn} \times \frac{V+1}{r-1} + \frac{n-m-1}{mn} \times \frac{V+1}{r.^m\overline{r-1}} + \frac{\overline{V+1}.2V}{mn.\overline{r-1}} - \frac{\overline{V+1}.2V}{mn.r^m\overline{r-1}}$. But $\frac{1}{r-1} - \frac{1}{r^m \overline{r-1}}$ is $= P$; therefore the above expression becomes $= V - \frac{V+1}{n} \times \overline{\overline{n-m-2V}-1 \times \frac{P}{m}} + 2V$. Q.E.D.

If the annuity be payable *half-yearly*, and $1 + \frac{r}{2}$ denote, as in the fifth Chapter of this volume, page 238, £1 increased by its interest for half a year, the series expressing the value of this annuity will be $= \frac{1}{2} \times \frac{\overline{m-\frac{1}{2}}.\overline{n-\frac{1}{2}}}{mn.\ 1+\frac{r}{2}} + \frac{1}{2} \times \frac{\overline{m-1}.\overline{n-1}}{mn.\ \overline{1+\frac{r}{2}}|^2} + \frac{1}{2} \times \frac{\overline{m-\frac{3}{2}}.\overline{n-\frac{3}{2}}}{mn.\ \overline{1+\frac{r}{2}}|^3} \cdots\cdots (2m)$. By proceeding in the same manner as in the foregoing theorem, and putting H to represent the value of an annuity certain, payable half yearly, for m years, the general rule in this case may be found $= V - \frac{V+\frac{1}{2}}{n} \times \overline{\overline{n-m-\frac{1}{2}-2V} \times \frac{H}{m}} + 2V$.

If the annuity be payable *quarterly*, and $1 + \frac{r}{4}$ denote £1 increased by its interest for three months, the series will be $= \frac{1}{4} \times \frac{\overline{m-\frac{1}{4}}.\overline{n-\frac{1}{4}}}{mn.\ 1+\frac{r}{4}} + \frac{1}{4} \times \frac{\overline{m-\frac{1}{2}}.\overline{n-\frac{1}{2}}}{mn.\ \overline{1+\frac{r}{4}}|^2} + \frac{1}{4} \times \frac{\overline{m-\frac{3}{4}}.\overline{n-\frac{3}{4}}}{mn.\overline{1+\frac{r}{4}}|^3}$ + &c.

$P = 21.195.\ n - m - 2V - 1 = -40.\ \overline{n - m - 2V - 1} \times \frac{P}{m} + 2V = 50 - 17.660 = 32.340$. And $V - \frac{V+1}{n} \times \overline{\overline{n - m - 2V - 1} \times \frac{P}{m} + 2V} = 25 - \frac{26}{59} \times 32.340 = 10.748$, the value of two joint lives whose ages are 27 and 38.

$+$ &c. - - - - - ($4m$), and its sum $= V - \frac{V + \frac{1}{4}}{n} \times \overline{\overline{n - m - \frac{1}{4} - 2V} \times \frac{Q}{m} + 2V}$, Q representing the value of an annuity certain payable *quarterly* for m years.—In like manner, if the annuity be payable *momently*, and $1 + \frac{1}{1000, \&c.}$ denote £1 increased by its interest for a moment, the general rule for determining the value of the annuity will be $= V - \frac{V + \frac{1}{1000, \&c.}}{n} \times \overline{\overline{n - m - \frac{1}{1000, \&c.} - 2V} \times \frac{M}{m} + 2V} = V - \frac{V}{n} \times \overline{\overline{n - m - 2V} \times \frac{M}{m} + 2V}$; M representing the value of an annuity certain payable momently for m years.

Supposing the ages of two lives to be 20 and 36 years, the value of an annuity at 4 *per cent.* during their joint continuance, and payable either yearly, half yearly, quarterly, or momently, will, by Mr. *De Moivre's* hypothesis, be 11.227...11.427...11.565...or 11.629. If their ages be 36 and 61, the values will be 7.448...7.673...7.793...or 7.901. If both their ages be 36, the values will be 10.394...10.600...10.703...or 10.808...and if both their ages be 61, the values will be 6.144...6.374...6.517...or 6.602.

By comparing the values of the equal joint lives, given above, with the values of the single lives of the same ages, computed in the fifth Chapter (p. 243 and 244), it appears that the differences in the former between annuities payable yearly and those which are payable half yearly, quarterly, or momently, are greater than the differences in the latter; and therefore that the addition to be made to an annuity on the longest of two lives, in consequence of its being payable at shorter intervals than a year, will be rather less than the addition to be made on this account, either to the single or the joint lives of the same ages.

M.

Note

Note (M). Chapter III. Page 197.

IT is plain that the purchaser of A's right, as stated in the first of the questions to which this note refers, cannot get into possession till the year when A and B shall be both dead; nor then, unless A happens to die *last*. Now, supposing the common complement of life n; the probability that A and B shall be *both* dead at the end of the *first* year, and A die last, is $\overline{1 - \frac{n-1}{n}} \times \overline{1 - \frac{n-1}{n}} \times \frac{1}{2} = \frac{1}{2} - \frac{n-1}{2n} - \frac{n-1}{2n} + \frac{\overline{n-1}|^2}{2n^2}$. In like manner, the probability that they shall be *both* dead at the end of the 2d, 3d, &c. years, and that A shall have died last is, $\frac{1}{2} - \frac{n-2}{2n} - \frac{n-2}{2n} + \frac{\overline{n-2}|^2}{2n^2}$; $\frac{1}{2} - \frac{n-3}{2n} - \frac{n-3}{2n} + \frac{\overline{n-3}|^2}{2n^2}$, &c. The *present* value, therefore, of the 1st, 2d, 3d, &c. rents of the reversionary estate is $\frac{1}{2r} - \frac{n-1}{2nr} - \frac{n-1}{2nr} + \frac{\overline{n-1}|^2}{2nr}$, $\frac{1}{2r^2} - \frac{n-2}{2nr^2} - \frac{n-2}{2nr^2} + \frac{\overline{n-2}|^2}{2n^2r^2}$, $\frac{1}{2r^3} - \frac{n-3}{2nr^3} - \frac{n-3}{2nr^3} + \frac{\overline{n-3}|^2}{2n^2r^3}$, &c. Supposing r to signify £1 increased by its interest for a year; and the estate to be £1 *per ann.* And the *sum* of these terms continued *in infinitum* is the value *required*. But $\frac{1}{2r} + \frac{1}{2r^2} + \frac{1}{2r^3}$, &c. is *half* the perpetuity. And $\frac{n-1}{2nr} + \frac{n-1}{2nr} - \frac{\overline{n-1}|^2}{2n^2r}$, &c. $\frac{n-2}{2nr^2} + \frac{n-2}{2nr^2} - \frac{\overline{n-2}|^2}{2n^2r^2} + \frac{n-3}{2nr^3} + \frac{n-3}{2nr^3} - \frac{\overline{n-3}|^2}{2n^2r^3}$, &c. is half the value of the *joint* lives, subtracted from *half* the sum of the values of the two *single* lives;

that is, *half* the value of the *longest* of the two lives. A similar demonstration may be applied to the other question (λ).

(λ). The purchaser of A's right, in the 2d Question, will get into possession in that year in which A either survives B or dies after him. The value of his expectation in the 1st year will be $= \frac{n-1}{nr} \times \overline{1-\frac{n-1}{n}} + \frac{1}{2r} \times \overline{1-\frac{n-1}{n}} \times \overline{1-\frac{n-1}{n}} = \frac{1}{2r} - \frac{\overline{n-1}|^{2}}{2nnr}$ .... In the 2d, 3d, 4th, &c. years, his expectation depending on the same events will be worth $\frac{1}{2r^{2}} - \frac{\overline{n-2}|^{2}}{2nnr^{2}}$, $\frac{1}{2r^{3}} - \frac{\overline{n-3}|^{2}}{2nnr^{3}}$, $\frac{1}{2r^{4}} - \frac{\overline{n-4}|^{3}}{2nnr^{4}}$, &c. The present value therefore of the 1st, 2d, 3d, &c. rents of the reversionary estate is $\frac{1}{2r} + \frac{1}{2r^{2}} + \frac{1}{2r^{3}} + \&c. - \frac{\overline{n-1}|^{2}}{2nnr} - \frac{\overline{n-2}|^{2}}{2nnr^{2}} - \frac{\overline{n-3}|^{2}}{2nnr^{3}} - \&c.$——If instead of an *estate*, the value of a given *sum* were required, it would, agreeable to the foregoing demonstrations, be expressed in the first case by $\frac{S.\overline{r-1}}{2r} \times \overline{V+BB-2B}$, and in the second case by $\frac{S.\overline{r-1}}{2r} \times \overline{V-BB}$ (V denoting the perpetuity, B the value of an annuity on the life of B, and BB the value of an annuity on two joint lives whose common age is that of B). The latter value therefore according to *De Moivre's* hypothesis, and in the particular case where the two lives are equal, exceeds the former value by $\frac{S\,\overline{r-1}}{r} \times \overline{B-BB}$.——That this is likewise true whatever be the decrements of life, or the ages of A and B, may be proved from the two Theorems in Note (O): For by the second of these theorems the value of S is $= \frac{S}{2} \times \overline{\frac{\beta.\overline{F-AF}}{b} - \frac{c.\overline{P-AP}}{br} - \frac{\overline{r-1}.\overline{B-AB}}{r}}$, and by the first it is $= \frac{S}{2} \times \overline{\frac{\beta.\overline{F-AF}}{b} - \frac{c.\overline{P-AP}}{br} + \frac{\overline{r-1}.\overline{B-AB}}{r}}$; from which it appears that the latter reversion exceeds the former by $\frac{S.\overline{r-1}}{r} \times \overline{B-AB}$, and consequently that the difference between them will be the same in all cases. M.

Note

Note (N). Chapter IV. Page 204.

LET r be £1 increased by its interest for one year.

Let S represent any given interval of time, or number of years, during which the decrements of life in a table of observations continue equal.

a the number of the living in the table at the beginning of the first year of that interval.

b the number of the living in the table at the beginning of the year immediately following the same interval.

P the value of an annuity certain for S years.

p the value of £1 due at the end of S years.

Q the value, in Table VI. in the next volume, of an annuity for the life of a person whose age wants S years of 86.

N the value, in strict agreement with the given table of observations, of an annuity on the life of a person whose age is S years greater than the age at which the interval of equal decrements begins. Then,

$Q + \frac{b}{a} \times \overline{P - Q}$ will be the value, according to the table of observations, of an annuity for S years, on a life of the same age with that at which the interval of equal decrements begins. And

$Q + \frac{b}{a} \times \overline{P - Q + pN}$ will be the value of an annuity on the whole duration of that life.

When S represents *one year*, Q vanishes, and the last expression becomes $\frac{b}{ar} \times \overline{1 + N}$; which is the rule for finding, from the value given of any life. the value of a life one year younger (μ).

In

(μ) The value of an annuity payable *half-yearly* during any life (A), may be deduced from the value of the same annuity during a life (B), one year younger than A, with nearly

In like manner, supposing G to signify the value of two given joint lives by any table of ob-

nearly as much ease as the values of annuities payable *yearly* are deduced. Let b represent the number of persons living in the Table at the age of B, and c, d, e, f, &c. the number living at the end of the 1st, 2d, 3d, &c. years from the age of B. Let r represent the interest of £1 for a year, and $p = \overline{1 + \frac{r}{2}}$; then will the value of the annuity be $= \frac{b+c}{4bp} + \frac{c}{2bp^2} + \frac{c+d}{4bp^3} + \frac{d}{4bp^4} + \frac{d+e}{4bp^5} +$ &c. which may be found $= \frac{1}{4p} + \frac{c}{4b} \times \overline{\frac{1}{p} + \frac{2}{p^2} + \frac{1}{p^3}} + \frac{d}{4b} \times \overline{\frac{1}{p^3} + \frac{2}{p^4} + \frac{1}{p^5}} + \frac{e}{4b} \times \overline{\frac{1}{p^5} + \frac{2}{p^6} + \frac{1}{p^7}}$, &c. From this series, if the age of B be very old, the value of the life annuity will be obtained with little difficulty; and having this, the value of an annuity on a life one year younger may be derived from it in the following manner:—Let a denote the number of persons living at the age of (A), who is one year younger than B; then, since the series expressing the value of an annuity on the life of the latter is found above to be $= \frac{b+c}{4bp} + \frac{c}{2bp^2} + \frac{c+d}{4bp^3} +$ &c. the series expressing the value of an annuity on the life of the former will be $= \frac{a+b}{4ap} + \frac{b}{2ap^2} + \frac{b+c}{4ap^3} +$ &c. $= \frac{a+b}{4ap} + \frac{b}{ap^2} \times \overline{\frac{1}{2} + \frac{b+c}{4bp} + \frac{c}{2bp^2} + \frac{c+d}{4bp^3} + \text{\&c.}}$ Therefore if the value of the annuity on the life of B be called M, the value of the annuity on the life of A will be $= \frac{a+b}{4ap} + \frac{b}{ap^2} \times \frac{1}{2} + M$.

From this Theorem a table may be computed of the values of annuities payable *half yearly* on lives of all ages; and by proceeding in the same manner a general Theorem may be obtained for computing a table of the values of annuities payable *quarterly*. But the labour of forming a table of this kind will be rendered unnecessary, if we are possessed of the values payable *yearly*: for I have found that the differences between annuities payable half yearly and yearly are the same, whether those values be derived from the real probabilities of life and the preceding Theorems, or from M. *De Moivre's* hypothesis, and the Theorems in the 5th Chapter in this volume. M.

servations, *a* the living at the age of one of them, *c* the living at the age of the other, and *b* and *d* the numbers living at the two next younger ages, $\frac{a \times c}{b \times d \times r} \times \overline{1 + G}$ will be the value of two joint lives each one year younger than the former.

The method of calculating the values of lives from any given tables of observations, described at the end of the fourth Chapter in this volume, is founded entirely on these Theorems; and a distinct explanation of them has been given by Mr. *Morgan*, in the Second Section of the Second Chapter of his book on the Doctrine of Life-Annuities and Assurances.

The expressions $Q + \frac{b}{a} \times \overline{P - Q}$, and $Q + \frac{b}{a} \times \overline{P - Q} + pN$, with their investigation, may be found in p. 341, 3d edition of Mr. *De Moivre's Treatise of the Doctrine of Chances*(v). But it is necessary to observe, that the direction which Mr. *De Moivre* has given for finding the value of Q is wrong. In consequence of calculating agreeably to this direction, he gives the value of a life at the age of 42 by Dr. *Halley's* table greater than the value of the same life by his own hypothesis; whereas, it is evident that the probabilities of

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(v) The Solution of this theorem may be deduced in a manner different from that of Mr. *De Moivre* Let α be the number of persons dying annually in s years, while the decrements of life continue equal, then will the value of the annuity during this term be $= \frac{a - \alpha}{ar} + \frac{a - 2\alpha}{ar^2} + \frac{a - 3\alpha}{ar^3} \ldots + \frac{a - s\alpha}{ar^s} = \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \ldots . (s) - \frac{s\alpha}{a} \times \overline{\frac{1}{sr} + \frac{2}{sr^2} + \frac{3}{sr^3} \ldots ( )}$ But the first series is $= P$, and the second series is $= - \frac{s\alpha}{a} \times \overline{P - Q} =$ (since $a - b$ is $= s\alpha$) $- \frac{a - b}{a} \times \overline{P - Q}$, and therefore the value of the annuity during the first s years will be $= Q + \frac{b}{a} \times \overline{P - Q}$.... The value of the annuity after s years supposing m, n, o, p, q, &c. to denote the number of persons living in the table at the end of $\overline{s + 1}$, $\overline{s + 2}$, $\overline{s + 3}$, &c. years

years is $= \frac{m}{ar^{s+1}} + \frac{n}{ar^{s+2}} + \frac{o}{ar^{s+3}} + \&c. = \frac{b}{ar^s} \times \overline{\frac{[illegible]}{br} + \frac{n}{br^2} + \frac{o}{br^3} + \&c.} = \frac{b}{a} \times pN$. If this expression be added to the value of the annuity, found above, for the first s years, the whole value will be $= Q + \frac{b}{a} \times \overline{P - Q + pN}$. Q. E. D.

It is necessary to observe that the series $\frac{a-\alpha}{ar} + \frac{a-2\alpha}{ar^2}$, &c. supposes the annuity to be payable *yearly*, and therefore that $\frac{s\alpha}{a} \times \overline{\frac{1}{sr} + \frac{2}{sr^2} + \frac{3}{sr^3}}$, &c. expresses the difference, multiplied into $\frac{s\alpha}{a}$, between the values of an annuity certain for s years, and of an annuity payable yearly during the continuance of a life whose complement is s.——The latter of these values, denoted by Q, is given in the 6th Table in the next volume.—But Mr. *De Moivre* has deduced the value of Q from the fluxional quantity $\frac{\dot{z}}{n \cdot \overline{r-1}} - \frac{\dot{z}}{nr^2 \cdot \overline{r-1}}$, which, expressing the value of an annuity *secured upon land*, must necessarily be always greater than the series $\frac{s-1}{sr} + \frac{s-2}{sr^2} +$ &c.; for the one supposes the annuity to be payable to the last moment of existence, while the other makes no allowance for that part of the year which shall have elapsed between its commencement and the extinction of the life. This value of Q therefore is improperly applied to the foregoing Theorem, where the value of N, as well as the whole solution, is founded upon the principle of the annuity's being payable only at the conclusion of each year, provided the life shall continue so long living after 42, being all along less in Dr. *Halley's* table than in the hypothesis, the value of the life must be also less.—The mathematical reader may easily satisfy himself, that the value of Q ought to be taken, as I have directed, from Table VI. in the next volume.

I cannot help adding here, that though the rules for finding, from the value given of any single or joint lives, the value of any single or joint lives one year younger, are an obvious corollary from the two expressions just mentioned, yet it is probable that Mr. *De Moivre* did not attend to them, or consider the facility which they give to calculations of this kind; for if he had, he would not probably have insisted so much as he has on his hypothesis of an equal decrement of life; much less would he, in order to obtain an easy method of calculation, have had recourse to that Second Hypothesis, which, in the fourth Chapter in this volume, has been shewn to be so very erroneous.

Mr. *Simpson* is, I believe, the first who has given these rules, in his Treatise on the Doctrine of Annuities and Reversions; but in his *Select Exercises*, p. 275, he has given a rule for approximating to the values of single lives, according to Dr. *Halley's* table, which must not be depended on, for I have found it half a year's purchase, and sometimes three-quarters of a year's purchase, wrong.

Note (O).

IN a note at the conclusion of the 3d Chapter *, Dr. *Price* refers to the end of this volume for more accurate solutions of his 11th and 12th Questions, which had been investigated by myself, and published in the 78th vol. of the Philosophical Transactions.—With the view of fulfilling his intentions in this respect, I shall here, in an abridged manner, insert the solutions to which he refers.

SOLUTION OF QUESTION XI. Page 40. Let a represent the number of persons living in the Table at the age of A, the younger of the two lives, a', a'', a''', &c. the decrements of life at the end of the 1st, 2d, 3d, &c. years from the age of A; b the number of persons living at the age of B, the older of the two lives, and c, d, e, f, &c. the number of persons living at the end of the 1st, 2d, 3d, &c. years from the age of B. Then will the value of S (the given sum), depending on the contingency of B's surviving A, be expressed by $\frac{S}{2ab} \times$

$$\overline{\frac{ca'}{r} + \frac{da''}{r^2} + \frac{ea'''}{r^3} + \text{\&c.}} + \frac{S}{2ab} \times \overline{\frac{ba'}{r} + \frac{ca''}{r^2} + \frac{da'''}{r^3} + \text{\&c.}}$$

$$= \frac{S}{2r} \times \overline{\frac{\beta r.\overline{F-AF}}{b} - \frac{c.\overline{P-AP}}{c} + \overline{r-1}.\overline{B-AB}};$$ F denoting a life one year younger, and P a life one year older than B; AF, AP, AB, the values of the joint lives of A and F, A and P, and A and B; and β the number of persons living in the Table at the age of F. But $\frac{\beta.F}{b}$ is $= \frac{1+B}{r}$ and $\frac{cP}{br} = B - \frac{c}{br}$. Hence if V represent the perpetuity, the

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the above expression may be found equal to $\frac{S}{2}$ $\times\ \overline{\frac{c.\overline{1+AP}}{br}+\frac{\overline{r-1}\,\overline{V-AB}}{r}-\frac{\beta.AF}{b}}$. Having now the value of the given sum payable on the contingency of B's surviving A, the value of the same sum payable on the contingency of A's surviving B is easily obtained; by subtracting the value found above from the whole value of the Reversion after the extinction of the joint lives of A and B.

SOLUTION OF QUESTION XII. page 43. Retaining the same symbols as in the preceding solution, the value of the sum S will in this case be $= \frac{S}{2ab}\times \overline{\frac{\overline{b-c}.a'}{r}+\frac{\overline{c-d}.a''}{r^2}+\frac{\overline{d-e}.a'''}{r^3}+\&c.}+\frac{S}{abr}\times\overline{\frac{a'.\overline{c-d}}{r}+\frac{\overline{a'+a''}.\overline{d-e}}{r^2}+\frac{\overline{a'+a''+a'''}.\overline{e-f}}{r^3}+\&c.} = \frac{S}{2r}\times \overline{\frac{\beta r.\overline{F-AF}-c.\overline{P-AP}}{b}-\overline{r-1}.\overline{B-AB}}$. or, substituting $\frac{1+B}{r}$ for $\frac{\beta F}{b}$ and $B-\frac{c}{br}$ for $\frac{cP}{br} = \frac{S}{2}\times \overline{\frac{c\times\overline{1+AP}}{br}+\frac{\overline{r-1}.\overline{V-2B-AB}}{r}-\frac{\beta AF}{b}}$. When the value of the reversion is required, depending on the contingency of A's having died after B, the foregoing value is to be subtracted from the whole value of the Reversion after the extinction of *both* lives. **M.**

Note (P.)

SECTION I.

IN the same note to which Dr. *Price* refers* for more accurate solutions of his 11th and 12th questions (and which have been given in the preceding pages), a further reference is made to the end of this volume, for rules which give in all cases correct values of sums payable on any survivorships between any three lives. These rules have been deduced by myself; and when the above note was written, it was my intention to have submitted the whole of them to Dr. *Price*, in order that he might use his own discretion in the manner of inserting them. But this is no longer possible, and I am induced therefore to give the greater part of them in this note. But being apprehensive of rendering my additions to this invaluable work much too long, I shall only insert the *rules*, referring the Reader for the *demonstrations* of them to the 79th, 81st, 84th, and 90th volumes of the Philosophical Transactions.

From the complicated nature of questions involving survivorships between three lives, it becomes necessary in their solution to have recourse to a great variety of symbols.—In order however to prevent repetition, the same symbols are uniformly made to denote the same quantities in all the following rules, and it may not be improper to begin with explaining them.

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* Page 199.

A. B. C. denote the value of an annuity on the respective lives of A, B, and C.

D. denotes the value of S on the contingency of C's surviving A (by Quest. XI. Note O).

E. denotes the same value on the contingency of B's surviving A, found by the same Question.

F. denotes the value of an annuity on a life one year younger than B.

G. denotes the value of the absolute Reversion of S after the death of A (by Quest. X. p. 32.)

H. denotes the value of an annuity on a life one year younger than A.

K. denotes the same value on a life one year younger than C.

L. denotes the value of an annuity on the longest of the three lives A, B, and C.

M. denotes the value of S, by the first Problem in this Note, on the contingency that A's life shall be the *first* that fails.

N. denotes the value of an annuity on a life one year older than A.

P. denotes the same value on a life one year older than B.

Q. denotes the value of S, by the 8th Problem, on the contingency of A *or* B, being *either* of them the *first* that fails.

R. denotes the value of S on the contingency of B's dying after A (by Quest. XII. Note O).

S. denotes the given sum.

T. denotes the value of an annuity on a life one year older than C.

V. denotes the perpetuity.

W. denotes the value of S on the contingency of C's dying after A (by Quest. XII. Note O).

and

α and a, denote the number of persons living in a table of observations at the ages of H and A.

β and b, denote the number of persons living at the ages of F and B.

κ and c, denote the number of persons living at the ages of K and C.

s, m, and d, denote the number of persons living at the end of the first year from the respective ages of A, B, and C.

r, denotes the value of £1 increased by its interest for a year.

The combinations of two or three of the several letters, A, B, C, F, H, &c. denote the values of annuities on the *joint* continuance of two or three of those respective lives.

PROBLEM I.

To determine the value of a given sum, payable if A should be the *first* that fails of the three lives A, B, and C.

SOLUTION.

When B or C are the oldest of the three lives the value of the Reversion will be = S into $\frac{\kappa}{3c} \times$

$$\overline{\frac{\beta.\mathrm{FK}-\mathrm{AFK}}{b}+\frac{\mathrm{BK}-\mathrm{ABK}}{2}}+\frac{\beta}{6b}\times\overline{\mathrm{FC}-\mathrm{AFC}}+\frac{\overline{r-1.}}{3r}\times\overline{\mathrm{BC}-\mathrm{ABC}}-\frac{m.\overline{\mathrm{PC}-\mathrm{APC}}}{6br}-\frac{d}{3c}\times\overline{\frac{\mathrm{BT}-\mathrm{ABT}}{2}+\frac{m.\overline{\mathrm{PT}-\mathrm{APT}}}{b}}.$$

When A is the oldest of the three lives the value will be $= S$ into $\frac{\beta}{3b} \times \overline{\frac{\alpha.\overline{HF + \frac{1}{2}HFC}}{a} - \overline{AF + \frac{1}{2}AFC}} + \frac{1}{6} \times \overline{\frac{\alpha.\overline{HB + 2HBC}}{a} - \overline{AB + 2ABC}} + \frac{1}{3r} \times \overline{\frac{s.\overline{BN - BNC}}{a} - \overline{AB - ABC}} + \frac{m}{6br} \times \overline{\frac{s.\overline{PN - PNC}}{a} - \overline{AP - APC}}$.

When the three lives are equal, the value will be $= \frac{S}{2} \times \frac{\overline{r-1.\overline{V-CCC}}}{r}$.

PROBLEM II.

To determine the value of a given sum, payable if A should be the *second* that fails of the three lives A, B, and C.

SOLUTION.

When the ages are *unequal*, the value of the Reversion will be $= D + E - 2M$.

When the ages are *equal*, its value will be $= \frac{S}{3} \times \frac{r-1}{r} \times \overline{V - \overline{3CC - 2CCC}}$.

PROBLEM III.

To determine the value of a given sum, payable on the death of A, if his life should be the *last* that fails of the three lives A, B, and C.

SOLUTION.

SOLUTION.

The value of the Reversion in this case will be either $G + M - \overline{D + E}$, or $\frac{S.\overline{r-1}.\overline{V-C}}{3r}$, according as the ages of the lives are unequal or equal.

PROBLEM IV.

To determine the value of a given sum, payable on the extinction of the lives of A and B, should they be the *first* that fail of the three lives A, B, and C.

SOLUTION.

Let Σ denote the value of S on the contingency of C's surviving B (by Quest. XI. Note O), and the general rule, when the lives are *unequal*, will be

$$= \Sigma \times \frac{S.x}{6a} \times \overline{HC - HBC} - \frac{S.x}{3} \times \overline{\frac{x.\overline{HK - HBK}}{2a} + \overline{AK - ABK}} - \frac{2S.\overline{r-1}}{3r} \times \overline{AC - ABC} - \frac{S.s}{6ar} \times \overline{NC - NBC} + \frac{S.d}{3ar} \times \overline{\overline{AT - ABT} + \frac{s.\overline{NT - NBT}}{2a}}.$$

------ If the three lives be *equal*, the Rule becomes

$$= \frac{S.\overline{r-1}}{3r} \times \overline{V - \overline{3CC - 2CCC}}.$$

PROBLEM V.

To find the value of a given sum, payable on the death of A, if his life should be the *first* or *second* that fails of the three lives A, B, and C.

SOLUTION.

SOLUTION.

The value of the Reversion, when the lives are *unequal*, will be $= D + E - M$.

When the lives are *equal*, it will be $= \frac{S.\overline{r-1}}{3r} \times \overline{2V - 3CC - CCC}$.

PROBLEM VI.

To find the value of a given sum, payable on the death of A, should his life be the *second* or *third* that fails of the three lives A, B, and C.

SOLUTION.

If the lives be *unequal*, the value of the Reversion will be $= G - M$.——If the three lives be *equal*, its value will be $= \frac{S.\overline{r-1}}{3r} \times \overline{2V - 3C - CCC}$.

PROBLEM VII.

To find the value of a given sum, payable on the death of A, should his life be the *first* or the *last* that fails of the three lives A, B, and C.

SOLUTION.

In this case the value of the Reversion will be $= G - \overline{D + E} + 2M$, if the lives be *unequal*, and $= \frac{S.\overline{r-1}}{3r} \times \overline{2V - 3C - 3CC + 2CCC}$, if the lives be *equal*.

PROBLEM

PROBLEM VIII.

To determine the value of a given sum, payable on the death of A or B, should *either* of them be the *first* that fails of the three lives A, B, and C.

SOLUTION.

Let Σ, as in Prob. IV. denote the value of S on the contingency of C's surviving B, and the value of the Reversion, *when C is the oldest of the three lives*, will be $=$ S into $\frac{x}{3c} \times \overline{\frac{\beta.\overline{FK - AFK}}{2b} + \overline{BK - ABK} - \frac{\beta.\overline{FC - AFC}}{6b} + \frac{2.\overline{r-1}.\overline{BC - ABC}}{3r} + \frac{m.\overline{PC - APC}}{6br}} - \frac{d}{3cr} \times \overline{\overline{BT - ABT} + \frac{m.\overline{PT - APT}}{2b}} + \Sigma$.----But if A be *the oldest*, the value will be $=$ S into $\frac{\overline{r-1}.\overline{V-AB}}{r} - \frac{\alpha}{3a} \times \overline{\frac{\beta.\overline{HF - HFC}}{b} + \frac{\overline{HB - HBC}}{2} - \frac{\beta.\overline{AF - AFC}}{6b}} + \frac{2.\overline{r-1}}{3r} \times \overline{AB - ABC} + \frac{m.\overline{AP - APC}}{6br} + \frac{s}{3ar} \times \overline{\frac{\overline{BN. - BNC}}{2} + \frac{m.\overline{PN - PNC}}{b}}$.---- And if the three lives be *equal*, the value will be $= \frac{2.S.\overline{r-1}}{3r} \times \overline{V - CCC}$.

PROBLEM IX.

To determine the value of a given sum, payable on the death of A or B, should *either* of them be the *second* that fails of the three lives A, B, and C.

SOLUTION.

SOLUTION.

When the lives are of *unequal* ages, the value of the Reversion will be $= \frac{S.\overline{r-1}.\overline{V-AB}}{r} + D + \Sigma - 2Q$ ---- (Σ denoting the same value as in Prob. IV. and VIII.) When the ages of the 3 lives are *equal*, the value will be $= \frac{2S.\overline{r-1}}{3r} \times \overline{V - \overline{3CC - 2CCC}}$.

PROBLEM X.

To find the value of a given sum, payable on the decease of B or C, should either of them be the *last* that fails of the three lives A, B, and C.

SOLUTION.

The value of the Reversion, when the lives are *unequal*, will be $= \frac{S.\overline{r-1}}{r} \times \overline{BC - ABC} + R + W - M$, and when the lives are all *equal*, it will be $= \frac{2S.\overline{r-1}}{3r} \times \overline{V - L}$.

PROBLEM XI.

To determine the value of a given sum, payable on the contingency of C's surviving B, provided the life of A shall be then extinct.

SOLUTION.

When either B *or* C *are the oldest of the three lives,* the value of the given sum will be $=$ S into $\frac{r}{6c} \times \overline{\frac{\beta.\overline{FK - FKC}}{b} - \overline{BK - ABK}} + \frac{\beta}{3b} \times \overline{FC - AFC} - \frac{r-1}{3r} \times \overline{BC - ABC} - \frac{m}{3br} \times \overline{PC - APC} + \frac{d}{6cr} \times \overline{\overline{BT - ABT} - \frac{m.\overline{PT - APT}}{b}}$.

When A *is the oldest of the three lives,* the value will be $= \Sigma - \frac{S.\varkappa}{3c} \times \overline{\frac{\alpha.\overline{HK - HBK}}{a} + \frac{\overline{AK - ABK}}{2}} - \frac{S.\alpha}{6a} \times \overline{HC - HBC} - \frac{\overline{r-1}}{3r} \times \overline{AC - ABC} + \frac{s}{6r} \times \overline{NC - NBC} + \frac{d}{3cr} \times \overline{\frac{\overline{AT - ABT}}{2} + \frac{s.\overline{NT - NBT}}{a}}$ (Σ denoting the same value as in Prob. IV. VIII. and IX.) ---- When the three lives are *equal*, the value of the Reversion will be $= \frac{S.\overline{r-1}}{6r} \times \overline{V - \overline{3CC - 2CCC}}$.

SECTION II.

THE remaining problems involve a contingency which makes it extremely difficult to give an exact solution of them. By the assistance, however, of the tables inserted at the end of this Note the difficulty may in a great measure be surmounted; but it is to be observed that the rules are unavoidably rendered more complicated, and therefore, in addition to the symbols already used, it has become necessary to have recourse to the following.

x, y, and z, denote the differences between the respective ages of A, B and C, and of the oldest person in the Table of Observations.

k, the number of persons living at the age of A at the end of x, y, or z years, according as A, B or C is the oldest life in the Problem.

p, the like number at the age of B.

q, the like number at the age of C.

δ, the probability (by Table II. at the end of this Note) that A dies *after* C.

ϵ, the probability (by the same Table) that C dies *after* B.

λ, the probability (by Table I. at the end of this Note) that C survives B.

μ, the probability (by Table II.) that C dies *after* A.

ξ, the probability (by the same Table) that B dies *after* C.

π, the probability (by the same Table) that B dies *after* A.

φ, the probability (by Table I.) that B survives A.

A^{y}, A^{z}, $\overline{AB}|^{y}$, $\overline{AC}|^{z}$, &c. B^{x}, B^{z}, $\overline{BC}^{x}$, &c. C^{x}, C^{y},

C^y, $\overline{CA|}^x$, &c. the values of annuities on the single or joint lives for *x*, *y*, *z* years.

$A^{\frac{1}{y}}$, $A^{\frac{1}{z}}$, $B^{\frac{1}{x}}$, $B^{\frac{1}{z}}$, $C^{\frac{1}{x}}$, $C^{\frac{1}{y}}$, &c. the values of annuities on single lives, *x*, *y*, or z years *older* than A, B, C, &c.

$\frac{1}{r^x}$, $\frac{1}{r^z}$, $\frac{1}{r^y}$, the value of £1 at the end of *x*, *y*, or z years.

PROBLEM XII.

To find the value of an annuity on the life of C after A, on the particular condition that A's life when it fails shall fail *before* the life of B.

SOLUTION.

If *C be the oldest of the three lives*, the value will be $= \frac{C - AC}{2} + \frac{BC - ABC}{2}$. If *A be the oldest*, the value will be $= \frac{C^x + \overline{BC}^x}{2} + \varphi . \overline{C - C^x} - \frac{AC + BAC}{2}$. If *B be the oldest*, the value will be $= \frac{C^y - \overline{AC|}^y}{2} + \pi . \overline{C - C^y} + \frac{BC - ABC}{2}$. And if the three lives be equal, the value will be $= \frac{C - CCC}{2}$.

PROBLEM XIII.

To determine the value of an annuity during the life of C after A, provided A should *survive* B.

SOLUTION.

SOLUTION.

The value when C *is the oldest of the three lives* will be $= \frac{C - AC}{2} - \frac{BC - ABC}{2}$; when A *is the oldest* it will be $= C - \frac{C^x + \overline{BC}|^x}{2} + \varphi . \overline{C - C^x} - \frac{AC - ABC}{2}$; when B *is the oldest* $= C - AC - \frac{C^y - \overline{AC}|^y}{2} - \frac{BC - ABC}{2} - \pi . \overline{C - C^y}$, and when the *three lives* are equal, the value will be $= \frac{C - 2 . CC + CCC}{2}$.

PROBLEM XIV.

To determine the value of a given sum payable on the death of A and C, provided B should *survive* one life in particular (A).

SOLUTION.

Let the value of an annuity of £1 depending on the contingency in Problem XII. be denoted by Y, and the value required, when the ages are *unequal*, will be $= E - \frac{S . \overline{r - 1}}{r} \times Y$.—But if the ages be *equal*, the value will be $= \frac{S . \overline{r - 1}}{r} \times \overline{V - CC - C - CCC}$.

PROBLEM XV.

To find the value of a given sum payable on the death of A and C, should B die *before* one life in particular (A).

SOLUTION.

SOLUTION.

Let the value of an annuity of £1 depending on the contingency in Problem XIII. be denoted by Z, let the value of the given sum after the death of A and B, provided B should die before A (found by the 2d Question in Note O) be represented by X, then will the value required be $= Z - \frac{S\,\overline{r-1}}{r} \times X$ if the ages of the three lives be *unequal*, or $= \frac{S.\overline{r-1}}{2r} \times \overline{V-L}$ if the ages be *equal*.

PROBLEM XVI.

To find the value of a given sum payable on the decease of B and C, should their lives be the *last* that shall fail of the three lives A, B, and C.

SOLUTION.

If B *and* C *are both of them older than* A, *and both of them nearly of the same age,* the value required will be $= S$ into $\frac{r-1}{r} \times \overline{\frac{V-ABC}{3} - \frac{B+C}{2} + BC - \frac{AB+AC}{2r}} + \frac{m}{2br} \times \overline{\overline{1+AP} - \frac{PC-APC}{3}} + \frac{d}{2cr} \times \overline{\overline{1-AT} - \frac{BT-ABT}{3} - \frac{4m.\overline{1+APT}}{3b}} - \frac{\beta.\overline{FC-AFC}}{3b} - \frac{\varkappa}{3c} \times \overline{\overline{BK}-ABK + \frac{\beta.AFK}{b}}$; *But if the age of* B *much exceeds that of* C the general rule will be more accurately expressed by S into $\frac{\overline{r-1}.\overline{BC-ABC}}{3r} + \frac{\varkappa}{3c} \times \overline{\frac{\beta.\overline{FK-AFK}}{b} - \overline{BK-ABK}}$

$$-\frac{\beta.\overline{FC-AFC}}{3b}+\frac{m}{2br}\times\frac{4d.\overline{PT-APT}}{3c}-\frac{\overline{PC-APC}}{3}-$$

$$\overline{P^y-\overline{AP}|^y}-\frac{d}{2cr}\times\frac{\overline{BT-ABT}}{3}+\overline{T^y-\overline{AT}|^y}+$$

$$\frac{\overline{B+C^y}-\overline{AB+AC}|^y}{2r}+\frac{\pi.q.\overline{r-1}}{cr^{y+1}}\times\overline{V-C_y}.$$——If

A *be the oldest of the three lives*, the general rule will be S into $\frac{\overline{r-1}.\overline{V-L}}{3r}+\frac{\overline{BC}|^x-C^x}{r}-$

$$\frac{\overline{r-1}.\overline{V+B^x+AB+AC}}{4r}+\frac{x}{2c}\times\frac{\alpha.\overline{HBK}}{3a}+\overline{K^x-\overline{BK}|^x}-$$

$$\frac{2.\overline{AK-ABK}}{3}+\frac{\alpha}{2a}\times\overline{HB}+\frac{4.\overline{HC-HBC}}{3}+\frac{s}{3ar}\times$$

$$\overline{NC-NBC}+\frac{d}{2cr}\times\overline{T^x-\overline{BT}|^x}+\frac{2s.\overline{T+NBT}}{3c}-$$

$$\frac{\overline{AT-ABT}}{3}+\frac{\pi.q.\overline{r-1}}{cr^{x+1}}\times\overline{V-C^x}+\mu+\frac{q}{c}\times\frac{p.\overline{r-1}}{br^{x+1}}$$

$\times\ \overline{V-B^x}$.——And *if the three lives be equal*, it will be simply $\frac{S.\overline{r-1}}{3r}\times\overline{V-L}$.

PROBLEM XVII.

To find the value of a given sum payable on the death of C, provided A should be the first, B the second, and C the third that shall fail of the three lives A, B, and C.

SOLUTION.

When C *is the oldest of the three lives*, the value of the given sum will be S into $\frac{x}{6c}\times$

$$\overline{\beta.\overline{FK-AFK}-\overline{BK}+\overline{ABK}}+\frac{\overline{r-1}.\overline{BC-ABC}}{6r}+\frac{m}{3br}$$

$\times \frac{d.\overline{PT-APT}}{c} - \overline{PC-APC} - \frac{\beta.\overline{FC-AFC}}{6b} + \frac{C-AC}{2r} + \frac{d}{2cr} \times \overline{\frac{BT-ABT}{3} - T - AT}$:— *When* B *is the oldest of the three lives,* the value will be S into $\frac{\overline{r-1}.\overline{BC-ABC}}{6r} + \frac{\varkappa}{6c} \times \overline{\frac{\beta.\overline{FK-AFK}}{b} - \overline{BK-ABK}} + \frac{m}{3br} \times \overline{\frac{d.\overline{PT-APT}}{c} - \overline{PC-APC} - \frac{\beta.\overline{FC-AFC}}{6b}} + \frac{C^y - \overline{AC}|^y}{2r} + \frac{d}{2rc} \times \overline{\frac{BT-ABT}{3} - T^y - \overline{AT}|^y} + \frac{\pi\, q.\overline{r-1}}{cry+1} \times \overline{V - C^{\frac{1}{y}}}$.——*When* A *is the oldest, and* B *the youngest of the three lives,* the general rule becomes S into $\frac{\overline{r-1}.\overline{V-L}}{6r} + \frac{\overline{BC}|^x - \overline{AC+C^x}}{2r} - \frac{\varkappa}{2c} \times \overline{BK + \frac{AK-ABK}{3}} + \frac{d}{2cr} \times \overline{\frac{AT-ABT}{3} + 1 + T^x - \overline{BT+BT}|^x} + \frac{s}{6ar} \times \overline{NC-NBC} + \frac{d.\overline{1+NBT}}{c} + \frac{\alpha}{3a} \times \overline{HC-HBC} + \frac{\varkappa.\overline{HBK}}{c} + \frac{\pi.q.\overline{r-1}}{crx+1} \times \overline{V - C^{\frac{1}{x}}}$.———But *when* A *is the oldest, and* C *the youngest of the three lives,* it becomes S into $\frac{r-1.}{6r} \times \overline{V - 3C^x - ABC} + \frac{\overline{r-1}.\overline{BC-AC}}{2r} - \frac{\beta}{2b} \times \overline{\frac{AF-AFC}{3} + \frac{\alpha\, HFC}{3a}} + \frac{\alpha}{2a} \times \frac{2.\overline{HB-HBC}}{3} + HC + \frac{s}{6ar} \times \overline{NB-NBC} + \frac{m\,\overline{1+NPC}}{b} - \frac{m}{2br} \times \overline{1+PC} + \frac{2.\overline{AP-APC}}{3} + \overline{\pi + \frac{p}{b}} \times \frac{\varkappa.\overline{r-1}.\overline{V.-C^{\frac{1}{x}}}}{crx+1} - \frac{p.\varkappa.\overline{r-1}.\overline{V-\overline{BC}|^{\frac{1}{x}}}}{2bcrx|+1}$.———And when *the three lives are equal,* the general rule is reduced to $\frac{S.\overline{r-1}.\overline{V-L}}{6r}$.

PROBLEM XVIII.

To determine the value of a given sum payable on the death of A or B, should either of them be the *first* or *second* that fails of the three lives A, B, and C.

SOLUTION.

The value of this reversion in all cases is $\frac{S.\overline{r-1}}{r} \times \overline{V-AB}$.

PROBLEM XIX.

To determine the value of a given sum payable on the decease of A or B, should either of them be the *second* or *third* that shall fail of the three A, B, and C.

SOLUTION.

The value of this reversion *when* A *and* B *are older than* C, and both of them nearly of the same age, will be S into $\frac{r-1}{r} \times \overline{V + ABC - \frac{\overline{A+B+C}}{2} + \frac{AC}{2r} - AB - \frac{\varkappa}{2c} \times \overline{AK + BK - 2ABK} + \frac{\beta}{2b} \times \overline{AF - AFC} + \frac{m}{2br} \times \overline{1 + AP} + \frac{d.\overline{PT - APT}}{c}}$. But if B *be considerably the oldest of the three lives,* the value will be S into $\frac{r-1}{r} \times \overline{V + ABC -}$

$\frac{\overline{A+B+BC}}{2}$

2

$$\frac{\overline{A^{y} + B + BC}}{2} + \frac{\overline{AC}|^{y}}{2r} - AB - \frac{x}{2c} \times \overline{\overline{AK}|^{y} + BK - 2BK}$$

$$+ \frac{\beta}{2b} \times \overline{AF - AFC} + \frac{m}{2br} \times \overline{1 + AP} + \frac{d.\overline{PT - APT}}{c}$$

$$+ \frac{\lambda k}{ar^{y}} \times \frac{\overline{r-1}.\overline{V - A^{\frac{1}{y}}}}{r}.$$

——————If C *be the oldest of the three lives,* the value of the reversion will be = S into

$$\frac{\overline{r-1} \times \overline{V - \frac{A^{z} + B^{z}}{2} + ABC}}{r} + \frac{BC + AC}{2r}$$

$$- AB + \frac{\beta.\overline{AF}|^{y} - AFC}{2b} + \frac{\alpha.\overline{HB}|^{y} - HBC}{2a} - \frac{s}{2ar} \times$$

$$\overline{1 + NC} + \frac{m}{br} \times \overline{\frac{s.\overline{1 + NPC}}{a} - \frac{\overline{1 + PC}}{2}} + \frac{p.\mu.\overline{r-1}.\overline{V - B^{\frac{1}{z}}}}{br^{z+1}}$$

$$+ \frac{\varsigma.k\,\overline{r-1}\,\overline{V - A^{\frac{1}{z}}}}{ar^{z+1}} + \frac{pk}{abr^{z+1}} \times \overline{1 + AB}|^{\frac{1}{z}}.$$

——————If *the three lives be equal,* the value will be = $\frac{S.\overline{r-1}}{r} \times \overline{V - C - CC + CCC}$.

PROBLEM XX.

To determine the value of a given sum after the decease of A or B, should either of them be the *first* or *last* that shall fail of the three lives A, B, and C.

SOLUTION.

If B *be the oldest of the three lives, the* value of the given sum will be S into $\frac{r-1}{r}$

$$\times \overline{V - ABC - \frac{A^{y} + B}{2} - \frac{\overline{AC}|^{y} + BC}{2r} + AB -}$$

$$\frac{\beta.\overline{AF - AFC}}{2b}$$

$\frac{\beta.\overline{AF - AFC}}{2b} + \frac{x}{2c} \times \overline{AK^{v} + BK - 2ABK} + \frac{m}{2br} \times \frac{d.\overline{1 + NPE}}{c} - \overline{A + AP} + \frac{\pi.k}{ar^{y}} \times \overline{r - 1.V - A^{\frac{1}{2}}}$.——

If C. *be the oldest of the three lives,* the value will be S into $\frac{r-1}{r} \times \overline{V - ABC - \frac{1}{2}A^{2} + B} - \frac{\beta.\overline{AF^{2} - AFC}}{2b} - \frac{\alpha.\overline{HH - HFC}}{2a} + \frac{m}{br} \times \frac{\overline{1 + PC}}{2} - \frac{s.\overline{1 + NPC}}{a} + \frac{\overline{AG + BG}}{2} + AB + \frac{s}{2ar} \times \overline{1 + NC +} \frac{\delta k.\overline{r - 1}}{ar^{z+1}} \times \overline{V - A^{\frac{1}{2}}} + \frac{\zeta p.\overline{r - 1}}{br^{z+1}} \times \overline{V - B^{\frac{1}{2}}} + \frac{\mu}{ab} \times \frac{\overline{r - 1}}{r^{z+1}} \times \overline{V - A^{\frac{1}{2}} + B^{\frac{1}{2}}} - \frac{kp}{abr^{z+1}} \times \overline{AB}^{\frac{1}{2}}$.—And

If the three lives be equal, the value will be $\frac{S.\overline{r - 1}}{r} \times \overline{V - C + CC - CCC}$.

PROBLEM XXI.

To find the value of a given sum payable on the death of A, should his life be the *first* or *second* that fails, and should B's life, if it fails, become extinct before the life of C.

SOLUTION.

The value required in this Problem is no more than " the value of a given sum on the death " of A, should C survive him," and therefore is obtained by the Solution of the first Question in Note O.

PROBLEM XXII.

To determine the value of a given sum payable on the death of A, should his life be the *second*, or *third* that fails, and should B's life when it fails become extinct before the life of C.

SOLUTION.

When B *is the oldest of the three lives, the* value of the reversion will be = S into $\frac{r-1}{6r}\times$ $\overline{2V - 3A^{y} - ABC} - \frac{\varkappa}{3c} + \overline{BK - ABK} + \frac{\beta\varkappa\ AFK}{6bc}$ $+ \frac{m}{3br}\times\overline{PC - APC} - \frac{d.\overline{1+APT}}{2c} + \frac{\beta}{6b}\times$ $\overline{FC - AFC + 3AF} - \frac{d}{6cr}\times\overline{BT - ABT} - \frac{\varkappa}{2c}\times$ $\overline{AK}|^{y} + \frac{\overline{AC}^{y} - AB}{2r} + \overline{\varepsilon + \frac{q}{c}} \times \frac{k.\overline{r-1}.\overline{V - A^{\frac{1}{y}}}}{ar^{y+1}}$.——

When C *is the oldest of the three lives*, the value will be = S into $\frac{r-1}{6r}\times\overline{2V-3A^{z}-ABC}-\frac{\varkappa}{3c}\times$ $\overline{BK-ABK}+\frac{\beta\varkappa}{6bc}+AFK+\frac{m}{3br}\times\overline{PC-APC-}$ $\overline{\frac{d.\overline{1+APT}}{2c}}-\frac{d}{6cr}\times\overline{BT-ABT}+\frac{\beta}{6b}\times\overline{FC-AFC}$ $\overline{+3AF}|^{z}-\frac{\varkappa}{2c}\times AK+\frac{AC-\overline{AB}|^{z}}{2}+\frac{\varepsilon k.\overline{r-1}.\overline{V-A^{\frac{1}{z}}}}{ar^{z+1}}$.

When A *is the oldest*, the value will be = S into $\frac{r-1}{6r}\times\overline{2V-3A-ABC}-\frac{\varkappa}{3c}\times\overline{AK-ABK}$ $\overline{-\frac{\alpha\varkappa.HBK}{3ac}}+\frac{\alpha}{6a}\times\overline{HC-HBC}-\frac{d}{6cr}\times$ $\overline{AT-ABT}+\frac{s.\overline{1+NBT}}{a}-\frac{s}{2ar}\times\frac{\overline{NC-NBC}}{3}-$ $\overline{1+NB}$

$-\overline{1+NB}+\frac{AC-AB}{2r}$; and when the *lives are equal*, the value will be $\frac{S.\overline{r-1}}{6r}\times\overline{2V-3C-CCC}$.

PROBLEM XXIII.

To determine the value of a given sum payable on the death of A, should his life be the *first* or *last* that shall fail of the three lives; and should B's life, if it fail, become extinct before the life of C.

SOLUTION.

If B *be the oldest of the three lives*, the value will be S into $\frac{\overline{r-1}}{2r}\times\overline{V-A^{z}-ABC}+\frac{AC}{2}-\frac{AB}{2r}+\frac{md}{2bcr}\times\overline{1+APT}-\frac{d}{2cr}\times\overline{1+AT+BT-ABT}-\frac{\beta\kappa}{2bc}\times AFK+\frac{\beta}{2b}\times\overline{AF+FC-AFC}+\overline{\varepsilon+\frac{q}{[illegible]}}\times\frac{k.\overline{r-1}.\overline{V-A}^{\frac{1}{y}}}{ar^{y+1}}-\frac{kq.\overline{r-1}.\overline{V-AC}|^{\frac{1}{y}}}{2acr^{y+1}}$. —— If C *be the oldest*, the value becomes S into $\frac{\overline{r-1}}{2r}\times\overline{V-A^{z}-ABC}+\frac{AC}{2}-\frac{\overline{AB}|^{z}}{2r}+\frac{md}{2bcr}\times\overline{1+APT}-\frac{d}{2cr}\times\overline{1+AT+BT-ABT}-\frac{\beta\kappa}{2bc}\times AFT+\frac{\beta}{2b}\times\overline{AF|^{z}+FC-AFC}+\frac{\varepsilon k.\overline{r-1}.\overline{V-A}^{\frac{1}{z}}}{ar^{z+1}}$. ——

If A *be the oldest*, it becomes S into $\frac{r-1}{2r}\times\overline{V-A-ABC}+\frac{AC}{2}-\frac{AB}{2r}+\frac{\alpha.HBC}{2a}-\frac{\kappa.AK}{2c}-\frac{d}{2cr}\times\overline{AT-ABT}+\frac{s}{2ar}\times\overline{1+NC-NBC}-$

$\frac{ds}{2acr} \times \overline{\mathrm{H} + \mathrm{NBT}}$:———And, if *the three lives be equal*, it becomes $= \frac{\mathrm{S}.\overline{r-1}}{2r} \times \overline{\mathrm{V} - \mathrm{C} + \mathrm{CC} - \mathrm{CCC}}$.

PROBLEM XXIV.

'To determine the value of a given sum payable on the death of A, B, and C, provided C shall die *after* one life in particular (A).

SOLUTION.

The value of the reversion in this case, when B *is the oldest of the three lives*, will be S into $\frac{r-1}{2r} \times \overline{\mathrm{BC} - \mathrm{B} - \mathrm{ABC} + \frac{\mathrm{BC}}{2} - \frac{\mathrm{AB}}{2}} + \frac{m}{2br} \times \overline{1 - \mathrm{AP} - \frac{d.\overline{1+\mathrm{APT}}}{c}} - \frac{x}{2c} \times \overline{\mathrm{BK} - \mathrm{ABK} + \mathrm{W}}$.—When C *is the oldest of the three lives*, the value will be S into $\frac{r-1}{2r} \times \overline{\mathrm{BC} - \mathrm{B} - \mathrm{ABC} + \frac{\mathrm{BC}}{2} - \frac{\mathrm{AB}}{2r}} + \frac{m}{2br} \times \overline{1 + \mathrm{AP} - \frac{d.\overline{1+\mathrm{APT}}}{c}} - \frac{x}{2c} \times \overline{\mathrm{BK} - \mathrm{ABK} + \mathrm{W}} + \frac{\mu p.\overline{r-1}.\overline{\mathrm{V}-\mathrm{B}}_{z}}{br^{z+1}}$.———When A *is the oldest of the three lives*, the value will be $=$ S into $\frac{r-1}{2r} \times \overline{\mathrm{V} - \mathrm{B} - 2\mathrm{C} + 2\mathrm{BC} + \mathrm{AC} - \mathrm{ABC}} + \frac{\mathrm{BC} - \mathrm{AB}}{2r} + \frac{x}{2a} \times \overline{\mathrm{HB} + \mathrm{HC} - \mathrm{HBC}} - \frac{x}{2c} \times \overline{\mathrm{BK} + \mathrm{AK} - \mathrm{ABK}} + \overline{\mu + \frac{1}{2}} \times \frac{p.\overline{r-1}.\overline{\mathrm{V}-\mathrm{B}}^{\frac{1}{z}}}{br^{z+1}} + \frac{\overline{b-p}.\overline{r-1}.\overline{\mathrm{V}-\mathrm{C}}^{\frac{1}{z}}}{bcr^{z+1}}$.——When *the three lives are of equal age*, the value will be $= \frac{\mathrm{S}.\overline{r-1}}{2r} \times \overline{\mathrm{V} - 3\mathrm{C} + 3\mathrm{CC} - \mathrm{CCC}}$, that is "half the reversion after the extinction of the three lives."

In the further pursuit of these enquiries I have discovered a simple method of approximating to the values in the preceding Problems; but it would be improper to enter more fully into the subject at present. I shall only observe here, that the solutions of those cases which involve three lives, and even those which involve two lives in the survivorship, being formerly deduced from an erroneous hypothesis, it was impossible to determine how far any approximations could be depended upon. By the assistance of the foregoing rules which have been derived from the real probabilities of life, and which I believe exhaust the subject in regard to three lives, this point may now be ascertained; though perhaps it may not often be adviseable to have recourse to *approximations*, when the *exact* values can be obtained with so little additional trouble.

The following Tables are inserted, as being necessary to the solution of the Problems in this Section. The demonstrations of the rules by which they have been computed may be seen in the 79th and 84th volumes of the Philosophical Transactions. M.

TABLE I.

Shewing the probabilities of survivorship between two persons of all ages, whose common difference of age is not less than ten years, computed from the Northampton Table of Observations.

Ten years difference.			Twenty years difference.			Thirty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
11— 1	.5859	.4141	21— 1	.5246	.4754	31— 1	.4823	.5177
12— 2	.5136	.4864	22— 2	.4433	.5566	32— 2	.3936	.6064
13— 3	.4823	.5177	23— 3	.4088	.5912	33— 3	.3557	.6443
14— 4	.4598	.5402	24— 4	.3900	.6100	34— 4	.3286	.6714
15— 5	.4470	.5530	25— 5	.3769	.6231	35— 5	.3135	.6865
16— 6	.4343	.5657	26— 6	.3640	.6360	36— 6	.2985	.7015
17— 7	.4253	.5747	27— 7	.3548	.6452	37— 7	.2883	.7117
18— 8	.4191	.5809	28— 8	.3484	.6516	38— 8	.2798	.7202
19— 9	.4160	.5840	29— 9	.3452	.6548	39— 9	.2753	.7247
20—10	.4153	.5847	30—10	.3441	.6559	40—10	.2733	.7267
21—11	.4158	.5842	31—11	.3440	.6560	41—11	.2724	.7276
22—12	.4168	.5832	32—12	.3442	.6558	42—12	.2718	.7282
23—13	.4178	.5822	33—13	.3444	.6556	43—13	.2714	.7286
24—14	.4190	.5810	34—14	.3447	.6553	44—14	.2711	.7289
25—15	.4202	.5798	35—15	.3451	.6549	45—15	.2708	.7292
26—16	.4216	.5784	36—16	.3456	.6544	46—16	.2706	.7294
27—17	.4226	.5774	37—17	.3458	.6542	47—17	.2701	.7299
28—18	.4232	.5768	38—18	.3454	.6546	48—18	.2690	.7310
29—19	.4234	.5766	39—19	.3444	.6556	49—19	.2672	.7328
30—20	.4231	.5769	40—20	.3429	.6571	50—20	.2650	.7350
31—21	.4222	.5778	41—21	.3408	.6592	51—21	.2621	.7379
32—22	.4209	.5791	42—22	.3384	.6616	52—22	.2589	.7411
33—23	.4197	.5803	43—23	.3359	.6641	53—23	.2556	.7444
34—24	.4183	.5817	44—24	.3335	.6665	54—24	.2522	.7478
35—25	.4170	.5830	45—25	.3309	.6691	55—25	.2487	.7513
36—26	.4156	.5844	46—26	.3283	.6717	56—26	.2452	.7548
37—27	.4141	.5859	47—27	.3256	.6744	57—27	.2416	.7584
38—28	.4126	.5874	48—28	.3228	.6772	58—28	.2379	.7621
39—29	.4110	.5890	49—29	.3199	.6801	59—29	.2341	.7659
40—30	.4094	.5906	50—30	.3170	.6830	60—30	.2302	.7698
41—31	.4078	.5922	51—31	.3143	.6857	61—31	.2263	.7737
42—32	.4063	.5937	52—32	.3116	.6884	62—32	.2223	.7777
43—33	.4049	.5951	53—33	.3088	.6912	63—33	.2181	.7819

Ten

Ten years difference.			Twenty years difference			Thirty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
44—34	.4034	.5966	54—34	.3060	.6940	64—34	.2138	.7862
45—35	.4019	.5981	55—35	.3032	.6968	65—35	.2093	.7907
46—36	.4004	.5996	56—36	.3003	.6997	66—36	.2047	.7953
47—37	.3988	.6012	57—37	.2973	.7027	67—37	.2001	.7999
48—38	.3971	.6029	58—38	.2943	.7057	68—38	.1954	.8046
49—39	.3954	.6046	59—39	.2913	.7087	69—39	.1906	.8094
50—40	.3938	.6062	60—40	.2882	.7118	70—40	.1859	.8141
51—41	.3923	.6077	61—41	.2849	.7151	71—41	.1810	.8190
52—42	.3907	.6093	62—42	.2814	.7186	72—42	.1758	.8242
53—43	3890	.6110	63—43	.2774	.7226	73—43	.1705	.8295
54—44	.3881	.6119	64—44	.2733	.7267	74—44	.1653	.8347
55—45	.3853	.6147	65—45	.2690	.7310	75—45	.1604	.8396
56—46	.3835	.6165	66—46	.2646	.7354	76—46	.1560	.8440
57—47	.3816	.6184	67—47	.2601	.7399	77—47	.1515	.8485
58—48	.3797	.6203	68—48	.2555	.7445	78—48	.1468	.8532
59—49	.3777	.6223	69—49	.2509	.7491	79—49	.1416	.8584
60—50	.3756	.6244	70—50	.2461	.7539	80—50	.1361	.8639
61—51	.3730	.6270	71—51	.2406	.7594	81—51	.1304	.8696
62—52	.3700	.6300	72—52	.2349	.7651	82—52	.1246	.8754
63—53	.3668	.6332	73—53	.2291	.7709	83—53	.1194	.8806
64—54	.3634	.6366	74—54	.2235	.7765	84—54	.1160	.8840
65—55	.3597	.6403	75—55	.2181	.7819	85—55	.1130	.8870
66—56	.3558	.6442	76—56	.2131	.7869	86—56	.1104	.8896
67—57	.3517	.6483	77—57	.2080	.7920	87—57	.1080	.8920
68—58	.3474	.6526	78—58	.2024	.7976	88—58	.1063	.8937
69—59	.3430	.6570	79—59	.1959	.8041	89—59	.1025	.8975
70—60	.3385	.6615	80—60	.1890	.8110	90—60	.0965	.9035
71—61	.3338	.6662	81—61	.1823	.8177	91—61	.0868	.9132
72—62	.3290	.6710	82—62	.1757	.8243	92—62	.0754	.9246
73—63	.3246	.6754	83—63	.1703	.8297	93—63	.0618	.9382
74—64	.3204	.6796	84—64	.1673	.8327	94—64	.0493	.9507
75—65	.3170	.6830	85—65	.1655	.8345	95—65	.0368	.9632
76—66	.3145	.6855	86—66	.1629	.8371	96—66	0258	.9742
77—67	.3120	.6880	87—67	.1638	.8362			
78—68	.3090	.6910	88—68	.1644	.8356			
79—69	.3047	.6953	89—69	.1623	.8377			
80—70	.3000	.7000	90—70	.1567	.8433			
81—71	.2958	.7042	91—71	.1450	.8550			
82—72	.2919	.7081	92—72	.1305	.8695			
83—73	2890	.7110	93—73	.1108	.8892			
84—74	.2902	7098	94—74	.0924	.9076			
85—75	.2917	.7083	95—75	.0717	.9283			
86—76	.2925	.7075	96—76	.0512	.9488			

Ten years difference.		
Ages.	Probabilities.	
87—77	.2942	.7058
88—78	.2994	.7006
89—79	.3029	.6971
90—80	.3022	.6978
91—81	.2897	.7103
92—82	.2720	.7280
93—83	.2420	.7580
94—84	.2049	.7951
95—85	.1606	.8394
96—86	.1173	.8827

Forty years difference.			Fifty years difference.			Sixty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
41— 1	.4357	.5643	51— 1	.3916	.6084	61— 1	.3494	.6506
42— 2	.3398	.6602	52— 2	.2887	.7113	62— 2	.2413	.7587
43— 3	.2985	.7015	53— 3	.2446	.7554	63— 3	.1952	.8018
44— 4	.2688	.7312	54— 4	.2128	.7872	64— 4	.1619	.8381
45— 5	.2520	.7480	55— 5	.1948	.8052	65— 5	.1428	.8572
46— 6	.2353	.7647	56— 6	.1768	.8232	66— 6	.1237	.8763
47— 7	.2230	.7770	57— 7	.1635	.8365	67— 7	.1092	.8908
48— 8	.2140	.7860	58— 8	.1535	.8465	68— 8	.0982	.9018
49— 9	.2086	.7914	59— 9	.1474	.8526	69— 9	.0912	.9088
50—10	.2059	.7941	60—10	.1442	.8558	70—10	.0873	.9127
51—11	.2045	.7955	61—11	.1422	.8578	71—11	.0846	.9154
52—12	.2035	.7965	62—12	.1406	.8594	72—12	.0825	.9175
53—13	.2026	.7974	63—13	.1390	.8610	73—13	.0805	.9195
54—14	.2018	.7982	64—14	.1376	.8624	74—14	.0788	.9212
55—15	.2011	.7989	65—15	.1362	.8638	75—15	.0775	.9225
56—16	.2005	.7995	66—16	.1350	.8650	76—16	.0768	.9232
57—17	.1999	.8001	67—17	.1335	.8665	77—17	.0759	.9241
58—18	.1981	.8019	68—18	.1314	.8686	78—18	.0743	.9257
59—19	.1959	.8041	69—19	.1286	.8714	79—19	.0719	.9281
60—20	.1931	.8069	70—20	.1253	.8747	80—20	.0690	.9310
61—21	.1896	.8104	71—21	.1211	.8789	81—21	.0655	.9345
62—22	.1855	.8145	72—22	.1165	.8835	82—22	.0616	.9384
63—23	.1812	.8188	73—23	.1120	.8880	83—23	.0581	.9419
64—24	.1768	.8232	74—24	.1075	.8925	84—24	.0556	.9444

Forty years difference.			Fifty years difference.			Sixty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
65—25	.1722	.8278	75—25	.1032	.8968	85—25	.0532	.9468
66—26	.1676	.8324	76—26	.0991	.9009	86—26	.0511	.9489
67—27	.1629	.8371	77—27	.0950	.9050	87—27	.0490	.9510
68—28	.1581	.8419	78—28	.0907	.9093	88—28	.0473	.9527
69—29	.1532	.8468	79—29	.0861	.9139	89—29	.0448	.9552
70—30	.1483	.8517	80—30	.0815	.9185	90—30	.0413	.9587
71—31	.1434	.8566	81—31	.0770	.9230	91—31	.0363	.9637
72—32	.1384	.8616	82—32	.0727	.9273	92—32	.0310	.9690
73—33	.1335	.8665	83—33	.0688	.9312	93—33	.0248	.9752
74—34	.1287	.8713	84—34	.0660	.9340	94—34	.0194	.9806
75—35	.1242	.8758	85—35	.0635	.9365	95—35	.0140	.9860
76—36	.1199	.8801	86—36	.0612	.9388	96—36	.0095	.9905
77—37	.1156	.8844	87—37	.0591	.9409			
78—38	.1112	.8888	88—38	.0574	.9426			
79—39	.1064	.8936	89—39	.0548	.9452			
80—40	.1021	.8979	90—40	.0511	.9489			
81—41	.0968	.9032	91—41	.0455	.9545			
82—42	.0919	.9081	92—42	.0392	.9608			
83—43	.0876	.9124	93—43	.0315	.9685			
84—44	.0846	.9154	94—44	.0247	.9753			
85—45	.0818	.9182	95—45	.0180	.9820			
86—46	.0795	.9205	96—46	.0123	.9877			
87—47	.0774	.9226						
88—48	.0759	.9241						
89—49	.0732	.9268						
90—50	.0690	.9310						
91—51	.0617	.9383						
92—52	.0533	.9467						
93—53	.0432	.9568						
94—54	.0342	.9658						
95—55	.0251	.9749						
96—56	.0173	.9827						

Seventy years difference.			Eighty years difference.			Ninety years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
71— 1	.3049	.6951	81— 1	.2495	.7505	91— 1	.1893	.8107
72— 2	.1945	.8055	82— 2	.1437	.8563	92— 2	.0918	.9082
73— 3	.1487	.8513	83— 3	.1034	.8966	93— 3	.0553	.9446
74— 4	.1159	.8841	84— 4	.0768	.9232	94— 4	.0312	.9688
75— 5	.0980	.9020	85— 5	.0634	.9366	95— 5	.0212	.9788
76— 6	.0798	.9202	86— 6	.0492	.9508	96— 6	.0115	.9885
77— 7	.0660	.9340	87— 7	.0385	.9615			
78— 8	.0553	.9447	88— 8	.0304	.9696			
79— 9	.0484	.9516	89— 9	.0250	.9750			
80—10	.0444	.9556	90—10	.0216	.9784			
81—11	.0418	.9582	91—11	.0186	.9814			
82—12	.0397	.9603	92—12	.0157	.9843			
83—13	.0380	.9620	93—13	.0125	.9875			
84—14	.0372	.9628	94—14	.0097	.9903			
85—15	.0368	.9632	95—15	.0070	.9930			
86—16	.0370	.9630	96—16	.0049	.9951			
87—17	.0373	.9627						
88—18	.0375	.9625						
89—19	.0367	.9633						
90—20	.0345	.9655						
91—21	.0309	.9691						
92—22	.0263	.9737						
93—23	.0210	.9790						
94—24	.0164	.9836						
95—25	.0118	.9882						
96—26	.0080	.9920						

TABLE II

Shewing the probability of one's life dying after another.*

Ten years difference.			Twenty years difference.			Thirty years difference.			Forty years difference.		
Ages.	Youngest.	Eldest.	Ages.	Youngest.	Eldest.	Ages.	Youngest.	Eldest.	Ages.	Youngest.	Eldest.
1 11	[illegible]	.5858	1 21	3885	.5241	1 31	3384	.4821	1 41	2908	.4355
2 12	4604	5136	2 22	4556	[illegible]	2 32	3934	3934	2 42	3355	.3596
3 13	4962	4822	3 23	4303	.4086	3 33	.4155	[illegible]	3 43	[illegible]	2983
4 14	[illegible]	.4597	4 24	4974	3598	4 34	.4307	.3284	4 44	3612	.2686
5 15	5297	4460	5 25	5028	3707	5 35	4382	.3133	5 45	3694	.2518
6 16	5112	4347	6 26	5120	.3538	6 36	4456	2983	6 46	3746	.2351
7 17	5561	4252	7 27	[illegible]	.3516	7 37	4498	2881	7 47	3777	.2228
8 18	5559	4190	8 28	5225	3482	8 38	[illegible]	2796	8 48	3791	.2138
9 19	5580	4159	9 29	5257	3450	9 39	[illegible]	2751	9 49	.3788	.2084
10 20	5591	4152	10 30	.5232	.3439	10 40	4532	.2731	10 50	3772	2057
11 21	[illegible]	4157	11 31	.5223	3458	11 41	[illegible]	2722	11 51	.3747	.2043
12 22	5571	4167	12 32	5209	3440	12 42	4497	.2716	12 52	3719	.2033
13 23	5558	4178	13 33	5187	.3440	13 43	4476	.2712	13 53	[illegible]	.2024
14 24	5511	4189	14 34	5179	3445	14 44	4453	.2709	14 54	[illegible]	2016
15 25	5530	4201	15 35	5162	3449	15 45	4430	.2706	15 55	3626	.2009
16 26	[illegible]	4215	16 36	5144	3451	16 46	.4405	.2704	16 56	3591	2003
17 27	[illegible]	4225	17 37	5128	3456	17 47	.4381	2699	17 57	3551	.1994
18 28	5460	4232	18 38	5117	.3462	18 48	.4360	2688	18 58	3523	.1978
19 29	[illegible]	4235	19 39	5109	3442	19 49	.4342	2670	19 59	3491	.1956
20 30	[illegible]	4230	20 40	5105	3427	20 50	.4325	.2648	20 60	3459	.1928
21 31	[illegible]	4221	21 41	5105	3406	21 51	.4312	2618	21 61	3428	.1893
22 32	5495	4209	22 42	5107	3382	22 52	4298	.2586	22 62	3399	.1852
23 33	5506	4196	23 43	.5110	3356	23 53	.4284	2553	23 63	[illegible]	.1809
24 34	[illegible]	4183	24 44	5110	3332	24 54	.4268	.2510	24 64	3330	1765
25 35	5524	4169	25 45	5112	3306	25 55	.4253	.2464	25 65	3307	.1719
26 36	5533	4155	26 46	.5112	3280	26 56	.4235	.2449	26 66	3274	1673
27 37	5543	4140	27 47	5113	.3253	27 57	.4212	.2413	27 67	.3238	.1626
28 38	.5553	.4125	28 48	5114	3225	28 58	.4199	.2376	28 68	3201	.1578

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* In this and the preceding Table the *certainty* of both lives becoming extinct is expressed by unity, this number being best suited to the solutions in the foregoing problem. It may not be improper to add, that both tables, though deduced from the decrements of life at Northampton, may be safely used, even when the values of the life annuities are derived from a different source, as the probabilities are very nearly the same, from whatever table of observations they are computed.

Ten years difference.				Twenty years differ.				Thirty years differ.				Forty years difference.			
Ages		Youngest	Eldest	Ages		Youngest	Eldest	Ages		Youngest	Eldest	Ages		Youngest	Eldest
29	39	.5562	.4110	29	49	.5115	.3196	29	59	.4179	.2338	29	69	.3164	.1528
30	40	.5578	.4091	30	50	.5115	.3167	30	60	.4150	.2299	30	70	.3122	.1479
31	41	.5583	.407[illegible]	31	51	.5112	.3140	31	61	.4136	.2260	31	71	.3077	.1430
32	42	.5592	.4063	32	52	.5108	.3113	32	62	.4112	.2220	32	72	.3029	.1380
33	43	.5601	.4048	33	53	.5104	.3085	33	63	.4089	.2177	33	73	.2977	.1331
34	44	.5608	.4034	34	54	.5099	.3057	34	64	.4066	.2131	34	74	.2921	.1282
35	45	.5615	.4022	35	55	.5092	.3029	35	65	.4037	.2089	35	75	.2858	.1237
36	46	.5625	.4003	36	56	.5086	.3000	36	66	.4009	.2043	36	76	.2788	.1194
37	47	.5635	.3986	37	57	.5072	.2969	37	67	.3978	.1997	37	77	.2715	.1150
38	48	.5646	.3968	38	58	.5071	.2939	38	68	.3945	.1950	38	78	.2637	.1106
39	49	.5655	.3951	39	59	.5051	.2909	39	69	.3910	.1902	39	79	.2559	.1057
40	50	.5661	.3936	40	60	.5049	.2878	40	70	[illegible]	.1854	40	80	.2470	.1014
41	51	.5669	.3920	41	61	.5038	.2845	41	71	.3829	.1805	41	81	.2381	.0960
42	52	.5676	.3901	42	62	.5026	.2810	42	72	.3785	.1753	42	82	.2286	.0910
43	53	.568	.3886	43	63	.5017	.2770	43	73	.3736	.1699	43	83	.2174	.0865
44	54	.5692	.3868	44	64	.5007	.2728	44	74	.3681	.1647	44	84	.2040	.0834
45	55	.5701	.3849	45	65	.4995	.2685	45	75	.3617	.1598	45	85	.1898	.0803
46	56	.5709	.3830	46	66	.4982	.2641	46	76	.3514	.1553	46	86	.1743	.0776
47	57	.5708	.3811	47	67	.4967	.2596	47	77	.3465	.1508	47	87	.1570	.0751
48	58	.5723	.3792	48	68	.4950	.2550	48	78	.338[illegible]	.1460	48	88	.1393	.0731
49	59	.5729	.3775	49	69	.4945	.2503	49	79	.3308	.1407	49	89	.1214	.0700
50	60	.5737	.3751	50	70	.4907	.2455	50	80	.3207	.1351	50	90	.1032	.0649
51	61	.5748	.3725	51	71	.4885	.2400	51	81	.3105	.1293	51	91	.0865	.0569
52	62	.5762	.369	52	72	.4860	.2342	52	82	.2993	.1234	52	92	.0691	.0476
53	63	.5778	.3662	53	73	.4830	.2284	53	83	.2861	.1180	53	93	.0520	.0363
54	64	.5792	.3629	54	74	.4793	.2227	54	84	.2706	.1143	54	94	.0324	.0252
55	65	.5811	.3591	55	75	.4747	.2173	55	85	.2530	.1110	55	95	.0126	.0126
56	66	.5830	.3551	56	76	.4691	.2122	56	86	.2338	.1079				
57	67	.5848	.3511	57	77	.4628	.2070	57	87	.2127	.1049				
58	68	.5868	.3467	58	78	.4561	.2013	58	88	.1891	.1025				
59	69	.5886	.3423	59	79	.4194	.1947	59	89	.1657	.0979				

Ten years difference.			Twenty years differ.			Thirty years difference.		
Age	Youngest	Eldest	Age	Youngest	Eldest	Age	Youngest	Eldest
60 70	.5904	.3377	60 80	.1421	.1870	60 90	.1424	.0909
61 71	.5921	.3350	61 81	4.332	.1808	61 91	.1204	.0801
62 72	.5937	.3281	62 82	[illegible]	.1710	62 92	.0973	.0674
63 73	.5940	.3230	63 83	[illegible]	.1683	63 93	.0740	.0520
64 74	.5950	.3195	64 84	.3935	.1640	64 94	.0467	.0364
65 75	.5912	.3156	65 85	[illegible]	[illegible]	65 95	.0181	.0184
66 76	.5921	.3135	66 86	.3512	.1607			
67 77	.5896	.3100	67 87	.3260	[illegible]			
68 78	.5868	.3072	68 88	.2956	.1586			
69 79	.5848	.3030	69 89	.2651	[illegible]			
70 80	.5822	.2981	70 90	[illegible]	.1173			
71 81	.5781	.2930	71 91	[illegible]	[illegible]			
72 82	[illegible]	.2863	72 92	.1695	.1006			
73 83	.5610	.2800	73 93	.1335	[illegible]			
74 84	.5508	.2860	74 94	.0878	.0682			
75 85	.5312	.2871	75 95	.0361	.0361			
76 86	.5148	.2868						
77 87	.4912	.2809						
78 88	.4601	.2806						
79 89	.4267	[illegible]						
80 90	.3894	.2801						
81 91	.3541	.2066						
82 92	.3145	.2422						
83 93	.2590	.2072						
84 94	.1835	.1540						
85 95	.0827	.0827						

Fifty years difference.

Ages		Youngest	Eldest
1	51	.2419	.3914
2	52	.2761	.2885
3	53	.2879	.2444
4	54	.2954	.2120
5	55	.2979	.1946
6	56	.3005	.1766
7	57	.3011	.1633
8	58	.3014	.1533
9	59	.2998	.1472
10	60	.2972	.1440
11	61	.2940	.1420
12	62	.2906	.1404
13	63	.2870	.1388
14	64	.2833	.1373
15	65	.2793	.1359
16	66	.2751	.1347
17	67	.2707	.1332
18	68	.2662	.1311
19	69	.2617	.1283
20	70	.2570	.1250
21	71	.2524	.1208
22	72	.2476	.1162
23	73	.2424	.1116
24	74	.2369	.1071
25	75	.2308	.1028
26	76	.2242	.0987
27	77	.2174	.0945
28	78	.2103	.0902
29	79	.2031	.0856
30	80	.1956	.0809
31	81	.1876	.0763
32	82	.1788	.0720
33	83	.1692	.0680
34	84	.1580	.0650
35	85	.1460	.0624
36	86	.1333	.0598
37	87	.1198	.0574
38	88	.1052	.0553
39	89	.0909	.0523
40	90	.0770	.0481
41	91	.0641	.0420
42	92	.0509	.0350
43	93	.0380	.0265
44	94	.0285	.0182
45	95	.0090	.0090

Sixty years difference.

Ages.		Youngest	Eldest
1	61	.1957	.3191
2	62	.2184	.2411
3	63	.2245	.1950
4	64	.2276	.1617
5	65	.2275	.1426
6	66	.2275	.1235
7	67	.2266	.1096
8	68	.2251	.0980
9	69	.2226	.0910
10	70	.2194	.0870
11	71	.2156	.0843
12	72	.2114	.0822
13	73	.2070	.0802
14	74	.2029	.0785
15	75	.1978	.0768
16	76	[illegible]	.0764
17	77	.1845	.0755
18	78	.1780	.0739
19	79	.1713	.0714
20	80	.1643	.0685
21	81	.1569	.0649
22	82	.1490	.0610
23	83	.1405	.0574
24	84	.1305	.0548
25	85	.1201	.0523
26	86	.1091	.0490
27	87	.0975	.0476
28	88	.0851	.0456
29	89	.0731	.0427
30	90	.0615	.0389
31	91	.0510	.0335
32	92	.0402	.0277
33	93	.0298	.0209
34	94	.0183	.0143
35	95	.0059	.0059

Seventy years differ.

Ages		Youngest	Eldest
1	71	.1544	.3046
2	72	.1622	.1942
3	73	.1604	.1484
4	74	.1573	.1156
5	75	.1523	.0977
6	76	.1477	.0795
7	77	.1433	.0657
8	78	.1390	.0550
9	79	.1340	.0481
10	80	.1301	.0440
11	81	.1250	.0414
12	82	.1193	.0392
13	83	.1137	.0375
14	84	.1068	.0366
15	85	.0993	.0361
16	86	.0912	.0360
17	87	.0821	.0362
18	88	.0723	.0360
19	89	.0628	.0350
20	90	.0525	.0327
21	91	.0433	.0280
22	92	.0342	.0235
23	93	.0253	.0177
24	94	.0155	.0121
25	95	.0059	.0059

Eighty years differ.

Ages		Youngest	Eldest
1	81	.1294	.2185
2	82	.1189	.1426
3	83	.1042	.1027
4	84	.0901	.0756
5	85	.0768	.0626
6	86	.0649	.0484
7	87	.0546	.0377
8	88	.0457	.0294
9	89	.0382	.0239
10	90	.0318	.0203
11	91	.0274	.0158
12	92	.0204	.0140
13	93	.0149	.0105
14	94	.0091	.0071
15	95	.0035	.0035

Ninety years differ.

Ages		Youngest	Eldest.
1	91	.1098	.1803
2	92	.0759	.0844
3	93	.0510	.0480
4	94	.0291	.0234
5	95	.0110	.0110

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END OF VOLUME FIRST.

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